Using a nonlinear Landweber algorithm to reconstruct 1D permittivity range profiles from coherent microwave backscattering data

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Abstract
This report deals with an application of the Landweber iteration to solve a one-dimensional inverse problem of interest in nondestructive evaluation. This algorithm is based on a nonlinear data model, and is particularized on a specific discretization of the equation of electromagnetic scalar scattering from lossless dielectric materials.

1. Introduction

Inverse scattering is an important issue in imaging and nondestructive evaluation, since diffracting wavefields are sensitive to physical quantities that cannot be probed by other exploring radiations. In particular, sonic, seismic, and electromagnetic waves already proved to be good candidates for a number of relevant applications.

I am not going to review all the matter here. I only mention that the two main strategies used to invert scattering data have been to linearize the intrinsically nonlinear scattering equations, or to invert numerically the nonlinear equations or some of their higher-order approximations. In other words, we could classify the inversion strategies on the basis of the data model and the inversion algorithm: the two approaches mentioned result in adopting a linear data model and a linear algorithm, or a nonlinear data model and a nonlinear algorithm. I also proposed to adopt a linear model and a nonlinear inversion algorithm, based on analytical properties of the solution. The "linear" Landweber approach that resulted is a mixture of linear reconstruction and superresolution, which draws better results from a linear Born model without being as expensive as most fully nonlinear approaches [1,2]. Indeed, the projected Landweber method I used is a nonlinear algorithm that can be applied to linear space-varying data models.

The fully nonlinear approaches to inverse scattering are normally very expensive computationally, since they rely on optimizing complicated functionals, normally by iterative numerical algorithms. The objective functionals, in turn, contain a data fit contribution and, implicitly or explicitly, a regularization term that takes prior information into account. Normally, a certain degree of local smoothness is assumed to regularize the objective functional. The information I use in my Landweber approach, compact support and positivity, is normally available in small-scale microwave nondestructive diagnosis, and entails substantial analytical properties of the solution. The problem is now to be able to exploit the same information in a technique based on a nonlinear data model. The theoretical aspects of this approach, that is, the possible analytical implications of compact support and positivity on a solution to the nonlinear scattering problem, should be studied thoroughly. Experimentally, however, it is possible to assess an existing strategy that exploits prior information: the Landweber method for nonlinear inverse problems [3].

To test this technique against scattering data, we must first make it explicit for a specific problem. In the linear case, I derived explicitly the Landweber iteration from the linear data model and then implemented numerically the iterative relations, making use of the discrete Fourier transform. In the nonlinear case, an analytic derivation of the algorithm from the integral formulation of the scattering problem is difficult. For this reason, I first discretized the scalar scattering equations and then derived the iterative procedure to be implemented. To start with, I took into account the one-dimensional range profile inversion problem.

Though relatively simple, this problem is interesting for some diagnostic applications, as evidenced by recent literature [4,5]. On the other hand, once the features of a Landweber iteration have been evaluated in this reduced setting, it will be possible to extend it to multidimensional problems. Another reason to consider the one-dimensional problem is that it is closer to practical applications, since a measurement system for single-view monostatic data can be easily realized, as opposed to the multiview, multistatic systems normally needed for multidimensional problems. However, it is to be stressed that the details of the algorithm described here derive from a particular discretization of the problem; different problems or discretizations would thus result in different algorithms.

In what follows, I present the direct scattering problem in the configuration of interest (Section 2). In Section 3, I formulate the inverse scattering problem in operatorial form, and present its solution by the nonlinear Landweber method. In Section 4, the problem is discretized by a method of moments and, in Sections 5 and 6, the Landweber iteration is particularized for this problem. An evaluation of the computational complexity per iteration is given in Section 7.

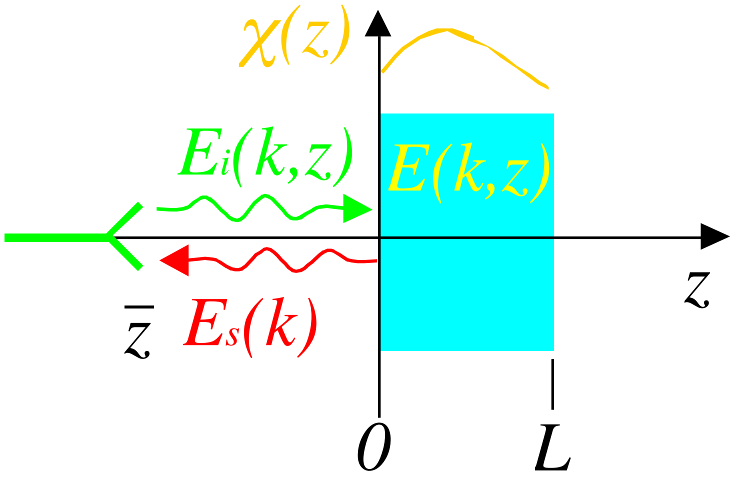

Figure 1: Illumination - measurement geometry of monostatic single-view scattering.

2. The one-dimensional direct scattering problem

Let us assume that a frequency-swept microwave source illuminates normally a lossless dielectric wall immersed in free space (see Figure 1).

The source produces a uniform scalar plane-wave incident field, $E_i(k,z)$, whose phasorial value is known as a function of the free-space wavenumber $k=2\pi f/c$ and the range coordinate z. Let us assume that the wall can be described by a dielectric contrast, $\chi(z)$, that is only a function of z and vanishes for $z \notin [0, L]$. Note that, since the material is lossless, $\chi(z)$ is real and does not depend on the wavenumber, that is, on the working frequency. The microwave source also acts as the sensor, and receives the backscattered field $E_s(k)$, which is measured for a discrete set of frequencies in magnitude and phase, taking $E_i(k,\bar{z})$ as the reference signal.

Assuming that $\chi(z)$ and $E_i(k,z)$ are known, the scattered field at $\bar{z}$ is the solution to the so-called external direct problem : for each value of k, we have

$$E_s(k) = -j\frac{k}{2}e^{jk\bar{z}}\int_0^L e^{-jkz'}\chi(z')E(k,z')dz' \tag{1}$$

where $j=\sqrt{-1}$, $E(k,z)$ is the total field inside the object, for wavenumber k and depth $z \in [0,L]$. In its turn, $E(k,z)$ is determined by the incident field and the contrast function, and is the solution to the so-called internal direct problem: for each k and for each $z \in [0,L]$, we have

$$E(k,z)=E_i(k,z)-j\frac{k}{2}\int_0^L e^{-jk|z-z'|}\chi(z')E(k,z')dz' \qquad (2)$$

Equation (2) implicitly defines an operator through which function $\chi(z)$ determines the values of the total field. Let us rewrite it as follows:

$$E_i(k,z)=E(k,z)+j\frac{k}{2}\int_0^L e^{-jk|z-z'|}\chi(z')E(k,z')dz' \qquad (3)$$

Let us assume that $\chi(z)$ belongs to the Hilbert space $L^2(0,L)$. In operatorial notation, we can write

$$E_i(k,z)=F_1^{-1}[\chi(z),E(k,z)] \qquad (4)$$

where the incident field E_i and the contrast function χ are known, and operator F_1^{-1} is defined by Equation (3). Inverting (4), we obtain again Equation (2) in operatorial notation:

$$E(k,z)=F_1[k,\chi(z)] \qquad (5)$$

This is an explicit relationship only formally, since, as already said, operator F_1 is defined implicitly by either Equation (2) or (3). In any case, Equation (5) formally denotes the solution to the internal direct scattering problem. Substituting (5) in (1), we obtain

$$E_s(k)=-j\frac{k}{2}e^{jk\bar{z}}\int_0^L e^{-jkz'}\chi(z')F_1(k,\chi(z'))dz' \equiv F[k,\chi(z)] \qquad (6)$$

Thus, operator F relates the contrast function to the scattered field at the sensor location, the incident field being known, and represents the solution to the external direct scattering problem.

3. The nonlinear 1D inverse scattering problem and its Landweber solution

Let us suppose now that the incident field is known and the scattered field for a number N_f of working frequencies is measured with a certain error. What is relevant to diagnostic applications is to derive an estimate of $\chi(z)$. In other words, from knowledge of $E_s(k)$ for N_f values of k, we should estimate $\chi(z)$ by inverting operator F defined in (6). Apparently, F is a nonlinear operator in χ, and is implicitly defined through operator F_1.

Although this formulation is approximated by neglecting the polarimetric nature of the fields involved, the inverse scattering problem is still difficult to solve. Nothing similar to the simple Fourier relationships that solve the problem with a linearized version of Equation (6) can be derived in this case [1].

Note that the formulation (6) is *continuous-continuous*, that is, the unknown is a continuous function of z and the data are a continuous function of k. In practice, as said, function $E_s(k)$ is only known for a discrete and finite set of values for k, that is, the problem is *continuous-discrete*. The most common approach to solve it has been to minimize the norm of $E_s - F[k,\chi(z)]$ in χ, for all values of k (see [6] for a review). Normally, this is done numerically, that is, by discretizing the integrals in (3) and (6). The problem is thus solved in a *discrete-discrete* setting. The inverse problem, however, is severely ill posed, and a regularization strategy must be employed to stabilize the solution. This is implemented by augmenting the objective functional to be optimized with a term related to prior information available, mostly enforcing some degree of smoothness in the solution. There is a class of analytical optimization algorithms that regularize the solution by stopping the iteration when the noise amplification prevents the solution from being further improved. The Landweber method is one of them, and the convergence features of its linear version have been studied very well [7]. Besides smoothness, any prior information can be enforced in the solution if it can be formalized as a condition of inclusion in a convex set [8,9]. This is the case with positivity, which has already been exploited in a successful application of the linear Landweber method to inverse scattering [1]. This report is aimed at exploring the possibility of applying the nonlinear version of the Landweber method (i.e., the method applicable to nonlinear data models) to our particular inverse scattering problem.

The nonlinear Landweber method to invert Equation (6) is an iterative procedure defined by an initial contrast estimate $\chi^{(0)}(z)$ and by the following update relationship [3,10]:

$$\chi^{(l+1)}(z) = \chi^{(l)}(z) - F'^{*}[k,\chi^{(l)}(z)][F(k,\chi^{(l)}) - E_s(k)] \tag{7}$$

where the update function is evaluated for all k, $F'(k,\chi)$ is the Fréchet derivative of operator F, and the asterisk denotes the adjoint operator. In this continuous-discrete setting, operator F maps $L^2(0,L)$ onto the set of the complex N_f-tuples defined, for each χ, by the N_f measured values of $E_s(k)$. The adjoint operator of the Fréchet derivative should thus map the set of the complex N_f-tuples onto $L^2(0,L)$.

4. Problem discretization

A way to develop Equation (7) is to discretize the continuous operator F defined in (5)-(6) and evaluate its Fréchet derivative, thus formulating the problem in a discrete-discrete setting. Let us divide the domain $[0,L]$ into N identical layers, with centers in z_i, $i = 1, \ldots N$ and thickness Δz (the maximum acceptable value for Δz should be about one fifth of the wavelength in the material). If the total field does not vary significantly within the single layer, a moment method [11,12] can be used to discretize the integrals in (2) and (1).

Before going on with our derivation, let us define the discrete equivalents of our original quantities: the elements of the (known) complex N-vectors

$$\mathbf{e}_n = [E_i(k_n,z_1),\ldots,E_i(k_n,z_N)]^T \qquad n = 1,\ldots N_f \tag{8}$$

are the phasorial values of the incident field at the centers of the N layers for the N_f different wavenumbers.

The complex N-vectors

$$\mathbf{t}_n = [E(k_n,z_1),\ldots,E(k_n,z_N)]^T \qquad n = 1,\ldots,N_f \tag{9}$$

are derived from the discretization of the total field in the domain $[0,L]$, as made in (8) for the incident field, and are obviously functions of the contrast χ.

Finally, the real N-vector

$$\chi = [\chi(z_1),\ldots,\chi(z_N)] \tag{10}$$

contains the contrast values at the centers of the layers. In the following, the m-th element of a generic vector $\mathbf{x}$ will be denoted by x_m. Similarly the (l,m)-th entry of a matrix $\mathbf{X}$ will be denoted by x_{lm}.
From positions (8)-(10), problem (3) can be discretized as

$$e_{n_m} = t_{n_m} + j\frac{k_n \Delta z}{2}\sum_{l=1}^{N} e^{-jk_n|z_m - z_l|}\chi_l t_{n_l} \qquad n = 1,\ldots N_f,\; m = 1,\ldots N \tag{11}$$

Let us define N_f matrices $\mathbf{A}_n(\chi)$ with generic element [11]

$$a_{n_{ml}}(\chi) = \delta_{ml} + j\frac{k_n \Delta z}{2} e^{-jk_n|z_m - z_l|}\chi_l \qquad n = 1,\ldots N_f; \qquad m,l = 1,\ldots N \tag{12}$$

where δ_{ml} is the Kronecker delta. Equation (11) can be written as

$$\mathbf{e}_n = \mathbf{A}_n(\chi)\mathbf{t}_n \qquad n = 1,\ldots N_f \tag{13}$$

from which we can explicitly state a discrete version of operator F_I defined by (4) and (5):

$$\mathbf{t}_n = \mathbf{A}_n^{-1}(\chi)\mathbf{e}_n \qquad n = 1,\ldots N_f \tag{14}$$

thus obtaining a discrete solution of the internal scattering problem, that is, we can build matrices $\mathbf{A}_n$ from χ, and then find the values of the total internal field by inverting them.
Let us now discretize the external problem (1):

$$E_s(k_n) = -j\frac{k_n \Delta z}{2} e^{jk_n\bar{z}}\sum_{m=1}^{N} e^{-jk_n z_m}\chi_m t_{n_m} \qquad n = 1,\ldots N_f \tag{15}$$

If we define the following diagonal matrices

$$\Lambda_n = diag\{e^{-jk_n z_m}\} \qquad n = 1,\ldots N_f \quad , \quad m = 1,\ldots N \tag{16}$$

and put Equation (15) in vector form, we get

$$E_s(k_n) = -j\frac{k_n \Delta z}{2} e^{jk_n\bar{z}}[\Lambda_n\chi]^T\mathbf{t}_n = -j\frac{k_n \Delta z}{2} e^{jk_n\bar{z}}\chi^T\Lambda_n\mathbf{t}_n \tag{17}$$

Finally, by exploiting (14) for $\mathbf{t}_n$, Equation (17) becomes

$$E_s(k_n) = -j\frac{k_n \Delta z}{2} e^{jk_n\bar{z}}\chi^T\Lambda_n\mathbf{A}_n^{-1}(\chi)\mathbf{e}_n \tag{18}$$

which is a solution to the direct scattering problem, providing the scattered field as a function of the known quantities χ and $\mathbf{E}$. A discrete version of operator F defined in (6) is

thus made explicit. In other words, for each wavenumber k (namely, for each index n) we have an explicit discrete operator that relates χ to the scattered field.
Let us now define a modified data vector **s**, whose generic element s_n is given by

$$s_n = j\frac{2}{k_n \Delta z}\exp\{-jk_n\bar{z}\}E_s(k_n) = \chi^T \Lambda_n \mathbf{A}_n^{-1}(\chi)\mathbf{e}_n \qquad n = 1,\dots N_f \tag{19}$$

where use has been made of Equation (18). In this new formulation, once all the elements of **s** have been evaluated, the inverse scattering problem becomes the inversion of a vector operator **F** that maps the set of real N-tuples onto the set of complex N_f-tuples.

$$\mathbf{s} = \mathbf{F}(\chi) \qquad \mathbf{s} \in C^{N_f}, \quad \chi \in \mathcal{R}^N \tag{20}$$

To implement iteration (7) in this discrete setting, we need the Fréchet derivative of operator **F**, that is, its Jacobian matrix. The adjoint of the Fréchet derivative will be the conjugate transpose of the Jacobian matrix. Each component of operator **F** is given by (19) for some n. The Jacobian matrix **J** is a complex $N_f \times N$ matrix whose nm-th entry is

$$J_{nm}(\chi) = \frac{\partial F_n(\chi)}{\partial \chi_m} \qquad n = 1,\dots N_f, \quad m = 1,\dots N \tag{21}$$

The development of the numerical method is continued in Sections 5 and 6, where matrix **J** is evaluated and a discrete version of iteration (7) is derived, respectively.

5. Calculating the Jacobian of the direct operator

From (19), we get

$$s_n = \chi^T[\mathbf{A}_n(\chi)\Lambda_n^{-1}]^{-1}\mathbf{e}_n = \chi^T \mathbf{B}_n^{-1}(\chi)\mathbf{e}_n \qquad n = 1,\dots N_f \tag{22}$$

where any element of matrix $\mathbf{B}_n$ can be derived from (12) and (16):

$$b_{n_{il}}(\chi) = a_{n_{il}}(\chi)\lambda_{n_{ll}}^{-1} = \delta_{il}\exp\{jk_n z_l\} + j\frac{k_n \Delta z}{2}\exp\{-jk_n[|z_i - z_l| - z_l]\}\chi_l \tag{23}$$

The Jacobian matrix elements in (21) are obtained by differentiating (22) with respect to the elements of χ. Let us explicitate the scalar operations in Equation (22):

$$s_n = \sum_{i=1}^{N} \sum_{l=1}^{N} \chi_i b_{n_{il}}^{[-1]}(\chi) e_{n_l} \tag{24}$$

where $b_{n_{il}}^{[-1]}(\chi)$ is the il-th element of $\mathbf{B}_n^{-1}(\chi)$. From (21) and (24), the generic element of the Jacobian matrix is

$$J_{nm}(\chi) = \frac{\partial s_n(\chi)}{\partial \chi_m} = \sum_{i=1}^{N} \sum_{l=1}^{N} \left[\frac{\partial \chi_i}{\partial \chi_m} b_{n_{il}}^{[-1]}(\chi) + \chi_i \frac{\partial b_{n_{il}}^{[-1]}(\chi)}{\partial \chi_m}\right] e_{n_l} \tag{25}$$

The derivative $\frac{\partial \chi_i}{\partial \chi_m}$ is obviously δ_{im}. The derivative $\frac{\partial b_{n_{il}}^{[-1]}(\chi)}{\partial \chi_m}$ of the il-th element of $\mathbf{B}_n^{-1}$ with respect to χ_m is still to be evaluated (see [13], p. 62):

$$\frac{\partial b_{n_{il}}^{[-1]}(\chi)}{\partial \chi_m} = \left[\frac{\partial}{\partial \chi_m} \mathbf{B}_n^{-1}(\chi) \right]_{il} \tag{26}$$

By the chain rule, we have:

$$\frac{\partial}{\partial \chi_m} \mathbf{B}_n^{-1}(\chi) = \sum_{r=1}^{N} \sum_{s=1}^{N} \frac{\partial \mathbf{B}_n^{-1}(\chi)}{\partial b_{n_{rs}}} \frac{\partial b_{n_{rs}}(\chi)}{\partial \chi_m} = \sum_{r=1}^{N} \sum_{s=1}^{N} -\mathbf{B}_n^{-1}(\chi)\mathbf{E}_{rs}\mathbf{B}_n^{-1}(\chi) \frac{\partial b_{n_{rs}}(\chi)}{\partial \chi_m} \tag{27}$$

where $\mathbf{E}_{rs}$ is a matrix with all null entries, except for the rs-th entry, which is equal to 1 (see [13], p. 64, for the derivative of a matrix with respect to the elements of its inverse). All the quantities needed to evaluate matrix $\mathbf{J}$ can now be calculated. By exploiting (23) in (27), we have:

$$\begin{aligned} \frac{\partial}{\partial \chi_m} \mathbf{B}_n^{-1}(\chi) &= \sum_{r=1}^{N} \sum_{s=1}^{N} -\mathbf{B}_n^{-1}(\chi)\mathbf{E}_{rs}\mathbf{B}_n^{-1}(\chi)[\delta_{sm} j \frac{k_n \Delta z}{2} \exp\{-jk_n[|z_r - z_s| - z_s]\}] = \\ &= j\frac{k_n \Delta z}{2} \sum_{r=1}^{N} -\mathbf{B}_n^{-1}(\chi)\mathbf{E}_{rm}\mathbf{B}_n^{-1}(\chi) \exp\{-jk_n[|z_r - z_m| - z_m]\} \end{aligned} \tag{28}$$

Finally, from (25), (26) and (28), the nm-th entry of matrix $\mathbf{J}$ will be:

$$\begin{aligned} J_{nm}(\chi) &= \sum_{i=1}^{N} \sum_{l=1}^{N} \left[\delta_{im} b_{n_{il}}^{[-1]}(\chi) + \chi_i \left[j\frac{k_n \Delta z}{2} \sum_{r=1}^{N} -\mathbf{B}_n^{-1}(\chi)\mathbf{E}_{rm}\mathbf{B}_n^{-1}(\chi) \exp\{-jk_n[|z_r - z_m| - z_m]\} \right]_{il} \right] e_{n_l} \\ &= \sum_{l=1}^{N} b_{n_{ml}}^{[-1]}(\chi) e_{n_l} + j\frac{k_n \Delta z}{2} \sum_{i=1}^{N} \sum_{l=1}^{N} \chi_i \left[\sum_{r=1}^{N} -\mathbf{B}_n^{-1}(\chi)\mathbf{E}_{rm}\mathbf{B}_n^{-1}(\chi) \exp\{-jk_n[|z_r - z_m| - z_m]\} \right]_{il} e_{n_l} \end{aligned} \tag{29}$$

To express this relationship in vector form, let $\mathbf{b}_{nm}^{[-1]T}$ be the m-th row of matrix $\mathbf{B}_n^{-1}$, and $\mathbf{C}_{nm}$ the bracketed matrix in (29). We have

$$J_{nm}(\chi) = \mathbf{b}_{nm}^{[-1]T}(\chi)\mathbf{e}_n + j\frac{k_n \Delta z}{2} \chi^T \mathbf{C}_{nm}\mathbf{e}_n = \left[\mathbf{b}_{nm}^{[-1]T}(\chi) + j\frac{k_n \Delta z}{2} \chi^T \mathbf{C}_{nm} \right] \mathbf{e}_n \tag{30}$$

6. Explicit Landweber iteration

The iterative scheme in (7) now becomes

$$\chi^{(l+1)} = \chi^{(l)} - \mathbf{J}^*(\chi^{(l)}) \left\{ \left[\chi^{(l)T} \mathbf{B}_n^{-1}(\chi^{(l)}) \mathbf{e}_n \right]_{n=1,\dots N_f} - \mathbf{s} \right\} \tag{31}$$

where $\mathbf{J}^*$ is the transpose conjugate of the Jacobian matrix (30), the vector in brackets comes from (22), and $\mathbf{s}$ is the modified data vector defined in (19). It should be verified whether the adjoint Jacobian matrix maps the vector in braces in the set of real N-vectors. This is not immediate, since the current χ estimate could well be complex, but this would not extend our data model. Indeed, we assumed a real, frequency-independent contrast function χ. Modeling lossy materials by complex permittivities produces frequency-dependent, complex contrast functions, and this greatly increases the number of unknowns of the inverse problem. To find a reasonable solution, we should thus assume lossless materials, characterized by real contrast functions. If the background medium is the free space, then the contrast function is also nonnegative. Reality and nonnegativity are pieces of information that could be exploited to speed-up the convergence of iteration (31). Indeed, the sets of real and nonnegative functions are closed and convex, and to project an L^2 complex function onto these spaces amounts to retain its real part where positive and setting the function to zero elsewhere [9,14]. Since the map in (31) normally produces complex N-vectors, the projection operations mentioned above can be used to implement a projected Landweber iteration (see [7] for the linear version). In practice, each estimated contrast can be projected onto the constraint sets before continuing the iteration. This can be made at each iteration or, as shown in [1,2], periodically after some fixed number of iterations. It has been shown that the iterative projections can sensibly improve the convergence (semi-convergence, strictly [7]). Another possibility to speed-up convergence is to use a suitable relaxation coefficient in (31) (see [1]).

Strictly speaking, we should check the existence of a solution to our problem by verifying that the map (31) is nonexpansive. In other words, for any other vector ψ belonging to the same space as χ, the following relationship should be verified

$$\|\chi - \mathbf{J}^*(\chi)\left\{\left[\chi^T\mathbf{B}_n^{-1}(\chi)\mathbf{e}_n\right]_{n=1,\dots N_f} - \mathbf{s}\right\} - \psi + \mathbf{J}^*(\psi)\left\{\left[\psi^T\mathbf{B}_n^{-1}(\psi)\mathbf{e}_n\right]_{n=1,\dots N_f} - \mathbf{s}\right\}\| \le \|\chi - \psi\|$$

(32)

$$\|\chi - \psi - \left\{\mathbf{J}^*(\chi)\left\{\left[\chi^T\mathbf{B}_n^{-1}(\chi)\mathbf{e}_n\right]_{n=1,\dots N_f} - \mathbf{s}\right\} - \mathbf{J}^*(\psi)\left\{\left[\psi^T\mathbf{B}_n^{-1}(\psi)\mathbf{e}_n\right]_{n=1,\dots N_f} - \mathbf{s}\right\}\right\}\| \le \|\chi - \psi\|$$

The existence of a domain where relationship (32) is verified should be checked case by case, since it cannot be assured in general. In [3], it is argued that this check is virtually impossible in most practical cases. A local convergence condition is then proposed for a ball of specified radius containing the initial guess.

The Landweber iteration converges if the Fréchet derivative of the direct operator has norm smaller than 1 in a ball containing the solution, with center $\chi^{(o)}$ and radius ρ. In our case, it must exist a real ρ such that, for any χ with $\|\chi - \chi^{(o)}\| \le \rho$, it is $\|\mathbf{J}(\chi)\| \le 1$ for some norm. If the Euclidean norm is used, the square root of the largest eigenvalue of matrix $\mathbf{J}^*\mathbf{J}$ must be not larger than 1 in some neighborhood of $\chi^{(o)}$ containing the solution.

The Landweber method, however, is not scale invariant. This means that scaling the data $E_s(k)$ and operator F in the original problem (6) by a positive factor C, the convergence properties of iteration (7) may result significantly altered. In [10], it is proved that a suitably scaled version of the Landweber iteration is equivalent to a relaxed iteration, whose convergence is analyzed in an increasingly fine discrete setting. A Landweber iteration with a suitable relaxation coefficient is equivalent to a scaled problem whose convergence is assured. If $\mathcal{B}_\rho(\chi^{(0)})$ is a ball with center in $\chi^{(o)}$ and radius ρ, this relaxed iteration can be written as:

$$\chi^{(l+1)} = \chi^{(l)} - \omega \mathbf{J}^*(\chi^{(l)}) \left\{ \left[\chi^{(l)T} \mathbf{B}_n^{-1}(\chi^{(l)}) \mathbf{e}_n \right]_{n=1,\dots N_f} - \mathbf{s} \right\} \tag{33}$$

with

$$\omega \in \left] 0, \frac{1}{C^2} \right[\tag{34}$$

and

$$C := \sup\left\{ \| \mathbf{J}(\chi) \| : \chi \in \mathcal{B}_\rho(\chi^{(0)}) \right\} \tag{35}$$

where ρ is such that a solution χ^* to Equation (6) is contained in the ball $\mathcal{B}_{\rho/2}(\chi^{(0)})$.
A numerical experimentation could be devoted to find a suitable ω to make the iteration convergent, and then to assess the effect of the projection onto the spaces of real and nonnegative functions. Different kinds of simulated data could precede an experimental phase with real backscattering measurements.

7. Computational issues

With noisy data, the projected Landweber method for 2D linear inverse scattering [1,2] has proved to require a few tens of iterations to converge. An extended experimentation would be needed to see whether this is also the case with one-dimensional nonlinear inverse scattering. What we are able to do now is to check the computational complexity of a single iteration. Let us start, then, by examining iteration (33). The complex N_f-vector **s** is derived from the measurements. The real N-vector $\chi^{(l)}$ is the current contrast estimate and is calculated in the previous iteration. The N_f complex vectors $\mathbf{e}_n$, of size N, contain the values of the incident field, assumed known. The N_f complex matrices $\mathbf{B}_n$, of size $N \times N$, must be built and inverted at each iteration. The generic element of one of these matrices is calculated by Equation (23), which requires one complex multiplication at each update. At each iteration, then, building matrices $\mathbf{B}_n$ requires $N_f N^2$ operations. Inverting these matrices with a direct method entails $N_f N^3$ complex operations. Building the bracketed vector in (33) requires $N_f N(N+1)$ operations. Finally, building the vector in braces will require a number of operations of the order of $N_f N^3$. Now we still need to evaluate the elements of the Jacobian matrix, as in (30). What we need is essentially to evaluate one matrix of the type $\mathbf{C}_{nm}$ for each entry of the Jacobian matrix. Let us rewrite the definition for $\mathbf{C}_{nm}$:

$$\mathbf{C}_{nm} = \left[\sum_{r=1}^{N} -\mathbf{B}_n^{-1} \mathbf{E}_{rm} \mathbf{B}_n^{-1} \exp\{-jk_n [|z_r - z_m| - z_m]\} \right] \tag{36}$$

where the inverse matrices $\mathbf{B}_n^{-1}$ have already been calculated, and the complex exponential term can be evaluated once for all, and stored in N_f matrices, each depending on the geometry of the problem alone. As already said, the entries of matrix $\mathbf{E}_{rm}$ are all zero, except the rm-th, which is unity. This means that all the rows of matrix $\mathbf{B}_n^{-1}\mathbf{E}_{rm}\mathbf{B}_n^{-1}$ are equal to the m-th row of matrix $\mathbf{B}_n^{-1}$ multiplied by the corresponding elements of its r-th column.

To build one of the matrices $\mathbf{B}_n^{-1}\mathbf{E}_{rm}\mathbf{B}_n^{-1}$, thus, N^2 complex multiplications will be needed. Each matrix should then be multiplied by the exponential term, and all this step will be repeated N times. Each element of $\mathbf{C}_{nm}$ thus needs $2N^3$ operations, to be repeated for all the $N_f N$ elements of the Jacobian matrix. To evaluate all the matrices $\mathbf{C}_{nm}$, $2N_f N^4$ complex operations will be needed. This is also the order of magnitude of the number of operations needed per iteration. As an example, for $N_f = 11$ and $N = 50$, hundreds of millions of complex multiplications are needed. For average processors, this would entail seconds of elapsed times per iteration, provided that all the fixed quantities can be stored in the computer's RAM. At each iteration, N_f complex matrices of size $N \times N$ should be stored. In the example mentioned, this entails 220 kB RAM, assuming all single-precision quantities. To avoid calculating the complex exponentials, we will need an additional 3D array of size $N_f \times N \times N$, which means an additional 220 kB. If we also want to store matrices $\mathbf{C}_{nm}$, we need room for $N_f N$ complex matrices of size $N \times N$, that is, in the same example, some 11 MB. This should not be a problem with the hardware available at present. What we do not know yet is how many iterations will be needed to have convergence. That the linear version in [1] typically needs just a few tens of iterations to reach the minimum reconstruction error does not mean that we can expect a comparable number of iterations in this case. This can encourage us to test this procedure, however.
Another issue to be considered is the possibility of simplifying some of the processing, by analyzing carefully the structure of the operations needed.

8. Conclusion

The Landweber iterative scheme for nonlinear data models has been made explicit for a discrete version of the 1D scalar inverse scattering problem. This algorithm could be very useful in several applications of nondestructive evaluation with diffracting probing wavefields. At present, we have no theoretical basis to assume that this algorithm can actually be useful. On one hand, its local convergence would be assured if we were able to evaluate the norm of the Jacobian matrix, and if the initial guess were assured to lie in a suitable neighborhood of the solution. This could be difficult to be verified theoretically. On the other hand, the semi-convergence conditions can be enforced empirically, by tuning the relaxation factor in (33) and by finding a strategy to choose an initial guess. Of course, these needs are obstacles to practical applicability.
The next step should be to actually implement the algorithm and experiment it against suitable measurement databases, possibly both simulated and real.

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