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Maria Girardi

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1. The paper presents an analytical approach for the study of the transverse oscillations of masonry columns.
2. The approach is first applied to free damped oscillations.
3. Then applications to forced oscillations in primary resonance on the column's first mode are shown.
4. Some examples are presented, comparing the analytical results with those obtained via the NOSA-ITACA code.
5. In the Appendix some remarks are included on the use of the averaged Lagrangian method.

Reviewer 1, first editorial comment.

On the dynamic behaviour of masonry beam–columns: an analytical approach

Maria Girardi^{a,*}

*^aIstituto di Scienza e Tecnologie dell'Informazione "A. Faedo", ISTI–CNR
Via G. Moruzzi 1, 56124 Pisa, Italy*

Abstract

The paper presents an analytical approach to the study of the transverse vibrations of masonry beam–columns. Starting with the constitutive equation for beams made of a masonry–like material and the averaged Lagrangian of the system, some explicit approximate solutions are found to the problem of free damped periodic oscillations and forced oscillations in the case of primary resonance on the beam's first mode. In particular, a set of equations is obtained that gives the modulation over time of the system's energy and of the fundamental frequency of the beam's response. The analytical results are compared to those obtained via the finite element code NOSA–ITACA, developed at ISTI–CNR.

Keywords: masonry–like materials, nonlinear dynamics, averaged Lagrangian method

1. Introduction

A constitutive model is proposed in [9], [36] for masonry–like materials with zero tensile strength and infinite compressive strength, where the constitutive equation for masonry–like materials [10], [11], [24], is specialized for masonry beams. The nonlinear elastic equation provided in [9], [36], which expresses the internal forces, normal force and bending moment, as functions of the generalized strains, stretching and change of curvature of the beam axis, has proven to be simple enough to enable some explicit calculations [9],

*Corresponding Author, Phone: +39 0503153075, Fax: +39 0503152040
Email address: Maria.Girardi@isti.cnr.it (Maria Girardi)

[16], [17], [36]. At the same time, its numerical implementation in the MADY code [18], [25], [27], represents a quick and effective way to assess the effects of the load's eccentricity on the static and dynamic behaviour of masonry columns, arches and towers.

For cyclic actions, this approach can furnish reasonable results for slender structures, for which the influence of shear forces on the dynamic equilibrium tends to decrease and the nonlinear behaviour is due essentially to the opening of cracks. **In fact, accurate modelling of the dynamic behaviour of masonry still represents an open problem. It is influenced by many parameters, involving the mechanical characteristics of the constituents materials, the construction techniques, the geometric characteristics, the kind of loading and the soil characteristics. Such complexity is extremely difficult to capture with a single model.**

Techniques for modeling masonry structures range from very complex micro-mechanical approaches [32], [28], to rigid block modelling for limit analysis [6],[19], to homogenisation techniques [3], [32], [33] and continuum models [1], [2], [13], [20].

With regard to the seismic vulnerability of masonry structures, the main instruments for engineers are based on vulnerability analyses derived from a statistic classification of the earthquake damage [22], [12] and on the use of numerical codes developed by defining macro-elements with a small number of degrees of freedom. Very few examples can be found of analytical approaches to the dynamic problem of masonry structures. (Rev1, comment 1)

In [17] the authors present an analytical study of the transverse vibrations of masonry beam-columns based on the constitutive equation described in [9], [36]. They limit themselves to considering free vibrations and obtain an explicit relation between the fundamental frequency of the beam and amplitude of the displacement. In the present paper the study is generalized in order to include damped and forced oscillations. In order to simplify computations, use is made of the averaged Lagrangian method proposed by G.B. Whitham to study the modulation of nonlinear dispersive waves [8], [34], [35]. This method reduces the problem to the study of a set of nonlinear differential equations – the so-called modulation equations – for some parameters of the problem, specifically energy and frequencies, which, if the nonconservative terms are small, can be considered slowly varying over time. The averaged Lagrangian method, whose use in the present context is jus-

tified in the Appendix, allows obtaining the modulation equations without the manipulations typical of other conventional methods based on series development, such as the multiple scales method [30], [31].

All results presented here have been obtained by assuming a unimodal expression for the beam's displacements and considering transverse vibrations only. With regard to the unimodal assumption, mainly depending on the frequency content of the excitation, some damage related to the presence of higher modes has been observed in many masonry towers and also modelled by numerical codes [4], [5]. However, the use of one single mode to describe the motion of slender masonry structures has generally proven to be able to capture some global damage patterns, such as the maximum displacements and the maximum compressive stresses along the structure. This hypothesis is also accepted by Italian regulations [12], provided the structure's geometry is regular. The interaction between longitudinal and transverse vibrations is also recognized to cause additional damage in slender masonry structures, especially for high values of the compressive stresses and in presence of vertical components of the dynamic excitation. (Second comment of Rev 1 and Rev 2)

To face the complex calculations involved in the analysis, we limit ourselves to considering opportune sets of initial conditions and harmonic transverse loads in primary resonance of the first mode, for which, provided that internal resonance phenomena do not occur [30], a unimodal solution is expected [21], [31]. Under these assumptions, the influence of the longitudinal vibrations on the transverse response of the beam can be neglected as well [26] and, provided that the normal force acting along the beam is known, the problems of the transverse and longitudinal vibrations of the beam-column can be dealt with separately. (Rev1, second comment and Rev2)

The paper is divided into three parts. In the first, the averaged Lagrangian method is presented and the modulation equations obtained for a broad class of nonlinear elastic materials. In the second, the method is applied to masonry-like materials, in the case of free damped and forced damped oscillations. Finally, the third part presents a parametric study, by varying on the one hand the slenderness and modal damping coefficient of the structure and, on the other, the forcing amplitude and frequency. All

results are compared with those obtained numerically via the finite element code NOSA–ITACA [24], <http://www.nosaitaca.it/>, developed at ISTI–CNR for static and dynamic equilibrium problems of masonry structures and constructions.

2. The averaged Lagrangian method

Let us consider a rectilinear beam with length l and rectangular cross section with height h and width b , subjected to a uniform axial force N and a transverse load per unit length q . **For the purposes of the present paper, we limit ourselves to considering solely the transverse vibrations of the beam. In this context, given a beam made of a nonlinear elastic material, its constitutive equation can be described by a function $M(\chi)$, where χ is the curvature of the beam and M the bending moment. (Rev2) This function is assumed to be continuously differentiable and its second derivative piecewise continuous.** Let us denote by E and ρ the Young's modulus and the density of the material, respectively, and by $J = bh^3/12$ the moment of inertia of the beam's section. Let the dissipative forces be modeled by a small viscous damping term in the form $\bar{C}v_t$, where v_t is the time derivative of the transverse displacement v , and \bar{C} is a positive constant (see Figure A.2).

In order to work with dimensionless quantities, if x and t are, respectively, the abscissa along the beam's axis and the time, we define

$$\xi = \frac{x}{l}, \quad \tau = \frac{t}{T_c}, \quad u = \frac{v}{l}, \quad \kappa = \chi l, \quad p = \frac{q T_c^2}{\rho b h l}, \quad C = \frac{\bar{C} T_c}{\rho b h} \quad (1)$$

with $T_c = l^2/c$ and $c = \sqrt{EJ/(\rho b h)}$ the elastic constant of the beam.

In the following we denote partial derivatives by both the compact notation, using subscripts, and the extended notation, using quotients, while primes denote total differentiation.

We assume the effects of both the shear strain and the rotary inertia to be negligible. Moreover, we limit ourselves to considering situations in which the flexural displacement $u(\xi, \tau)$ and its derivative $u_\xi(\xi, \tau)$ are small, so that we can neglect the effects of the axial force on the dynamic equilibrium of the beam and write

$$\kappa(\xi, \tau) = -u_{\xi\xi}(\xi, \tau). \quad (2)$$

In order to obtain small flexural displacements, we also consider the forcing term $p(\xi, \tau)$ to be small.

Under these hypotheses, putting

$$f(\kappa) = \frac{l}{EJ} M(\kappa/l), \quad (3)$$

the equation of motion is

$$\frac{\partial^2 u}{\partial \tau^2} - \frac{\partial^2 (f \circ \kappa)}{\partial \xi^2} = p - C \frac{\partial u}{\partial \tau}. \quad (4)$$

Now, let

$$L(u(\tau, \xi)) = \frac{1}{2} (u_\tau)^2 - F(-u_{\xi\xi}) \quad (5)$$

be the Lagrangian, where F is the primitive of f such that $F(0) = 0$. Thus, equation (4) can be written in the form

$$\frac{\partial}{\partial \tau} \left(\frac{\partial L}{\partial u_\tau} \right) - \frac{\partial^2}{\partial \xi^2} \left(\frac{\partial L}{\partial u_{\xi\xi}} \right) = p - C u_\tau. \quad (6)$$

We are looking for an approximate solutions to (6) of the form [8, 30, 31]

$$u(\xi, \tau) = \phi(\xi) \eta(\tau), \quad (7)$$

where $\phi(\xi)$ is a periodic function of the span of the beam, with

$$\int_0^1 \phi^2 d\xi = 1, \quad (8)$$

and $\eta(\tau) = U(\theta)$ is a periodic function with period 2π , phase θ and frequency $\omega = \theta_\tau$. In addition, U_θ is a periodic function with period 2π as well. It is a simple matter to verify [7], [17], that η satisfies the equation

$$\frac{\partial}{\partial \tau} \frac{\partial \bar{L}}{\partial \eta'} - \frac{\partial \bar{L}}{\partial \eta} = -2\mu\eta' + \bar{p} \quad (9)$$

where

$$\bar{L} = \int_0^1 L(\phi, \phi'', \eta, \eta') d\xi, \quad (10)$$

$$\mu = \int_0^1 C \phi^2 d\xi = \zeta\omega, \quad (11)$$

with ζ the damping ratio calculated under the hypothesis of modal damping [7], and

$$\bar{p} = \int_0^1 p \phi d\xi. \quad (12)$$

These hypotheses may be viewed as a rather strong restriction on the generality of the problem. However, provided that no internal resonance effects occur [31], the unimodal expression (7) has proven to work quite well in a large number of cases, such as the study of the primary resonance on the first mode of continuous nonlinear systems ([21], see also [30] for a **wide-ranging bibliographic review**) (**Rev1, comment 2**). In addition, its simplicity enables tackling the difficult calculations typically involved in nonlinear dynamical problems.

We now introduce the averaged Lagrangian [8],[34], [35]

$$\mathcal{L} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 L d\xi d\theta = \frac{1}{2\pi} \int_0^{2\pi} \bar{L}(U, \omega U_\theta) d\theta \quad (13)$$

where

$$\bar{L} = \bar{L}(U, \omega U_\theta) = \frac{1}{2} \omega^2 U_\theta^2 - V(U) \quad (14)$$

and

$$V(U) = \int_0^1 F(-\phi''\eta) d\xi \quad (15)$$

is a potential function.

For conservative systems, when $\mu = \bar{p} = 0$, a first integral of the equation of the motion (9) can be found in the form (see Appendix)

$$\frac{1}{2} \omega^2 U_\theta^2 + V(U) = a \quad (16)$$

where a is a constant representing the total energy of the system. Actually, provided that the nonconservative terms are sufficiently small, equation (16) can work for nonconservative systems as well. In this case a is no longer a constant but a slowly varying function of time, approximately constant during each loop of the oscillations. The same holds for frequency ω , which can be regarded as the "slow" variation of the phase θ over the time (see the Appendix). With the help of (14) and (16), the averaged Lagrangian (13)

becomes,

$$\mathcal{L} = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2} \omega^2 U_\theta^2 - V(U) \right) d\theta = \frac{\omega^2}{2\pi} \int_0^{2\pi} U_\theta^2 d\theta - a = \quad (17)$$

$$= \frac{\omega}{2\pi} \oint \sqrt{2(a - V)} dU - a = \mathcal{L}(a, \omega) \quad (18)$$

where integration with respect to θ and U is performed by assuming a and ω to be constant [35, 34]. The averaged Lagrangian \mathcal{L} depends only on the parameters a and ω , whose variation over time – generally referred to as “modulation” – is given by the Euler–Lagrange equations for \mathcal{L} :

$$\frac{\omega}{2\pi} \oint \frac{dU}{\sqrt{2(a - V(U))}} - 1 = -\frac{1}{2\pi} \int_0^{2\pi} \bar{p} U_a d\theta \quad (19)$$

$$\frac{a'}{2\pi} \oint \frac{dU}{\sqrt{2(a - V(U))}} = \mathcal{D} + \mathcal{P} \quad (20)$$

with

$$\mathcal{D} = \frac{1}{2\pi} \int_0^{2\pi} -2\mu\omega U_\theta^2 d\theta = -\oint \frac{\zeta\omega}{\pi} \sqrt{2(a - V)} dU \quad (21)$$

and

$$\mathcal{P} = \frac{1}{2\pi} \int_0^{2\pi} \bar{p} dU. \quad (22)$$

The righthand members of (19) and (20) represent the components of the vector of the generalized averaged forces acting upon the beam [29, 31]. Note that, for $\bar{p} = 0$, equation (19) is analogous to that obtained in [17] for free undamped oscillations by directly manipulating the equation of motion, while (20) is an energy balance equation, where the damping dissipation and the energy injection by the forcing term are taken into account. Provided that a potential $V(U)$ is known, the equations (19) and (20) work for a broad class of nonlinear elastic materials.

3. Application to masonry–like beams

Let us briefly recall the constitutive equation for masonry–like material with zero tensile strength and infinite compressive strength proposed in [9] and [36] **for a rectangular cross–sectional beam**.

Let χ be the curvature of the beam, ε **the infinitesimal axial strain (Rev2)**, M and N the bending moment and normal force, acting on the

beam's section. Under the classical Euler–Bernoulli hypothesis, we can deduce a relation between the generalized stresses N and M and the generalized strains ε and χ . To this end, given the subsets of the set E of all generalized strains (ε, χ)

$$\begin{aligned} E_1 &: \{(\varepsilon, \chi) \in E : 2\varepsilon \leq \chi h \leq -2\varepsilon, \quad \varepsilon \leq 0\}; \\ E_2 &: \{(\varepsilon, \chi) \in E : \chi h > 2|\varepsilon|, \quad \chi > 0\}; \\ E_3 &: \{(\varepsilon, \chi) \in E : \chi h < 2|\varepsilon|, \quad \chi < 0\}, \end{aligned} \quad (23)$$

we can obtain the relations:

$$\begin{aligned} (\varepsilon, \chi) \in E_1 &: N = Ebh\varepsilon, & M &= \frac{Ebh^3}{12}\chi; \\ (\varepsilon, \chi) \in E_2 &: N = -\frac{Eb(\chi h - 2\varepsilon)^2}{8\chi}, & M &= \frac{Eb(\varepsilon + \chi h)(\chi h - 2\varepsilon)^2}{24\chi^2}; \\ (\varepsilon, \chi) \in E_3 &: N = \frac{Eb(\chi h + 2\varepsilon)^2}{8\chi}, & M &= -\frac{Eb(\varepsilon - \chi h)(\chi h + 2\varepsilon)^2}{24\chi^2} \end{aligned} \quad (24)$$

with $N \leq 0$ and $\frac{Nh}{2} \leq M \leq -\frac{Nh}{2}$.

In consequence of the material's nonlinearity, both M and N depend on χ and ε , and the extensional and flexural problems are coupled.

So, if we presume to know, within a certain degree of approximation, the values of the normal force N along the structure, from (24) we can obtain a relation $M = M(\chi, N)$ that directly links the bending moment and the curvature. (Rev2) Thus, if we define

$$\chi_0 = -\frac{2N}{Ebh^2}, \quad (25)$$

where χ_0 is the curvature corresponding to the elastic limit, when a triangular stress distribution is reached on the section, the constitutive equation becomes

$$\frac{M(\chi)}{\rho bh} = \begin{cases} c^2\chi & \text{for } |\chi| \leq \chi_0, \\ c^2\chi_0 \text{Sign}(\chi)(3 - 2\sqrt{\frac{\chi_0}{|\chi|}}) & \text{for } |\chi| > \chi_0, \end{cases} \quad (26)$$

whose representation is given in Figure A.1.

The constitutive equation (26) assumes dimensionless form by putting

$$F(\kappa) = \begin{cases} \frac{1}{2}\kappa^2 & \text{for } \xi \leq \xi_0, \\ \kappa_0 \left(3|\kappa| - 4\sqrt{\kappa_0|\kappa|} \right) + \frac{3}{2}\kappa_0^2 & \text{for } \xi > \xi_0, \end{cases} \quad (27)$$

where F is the primitive of $f = Ml/(EJ)$ such that $F(0) = 0$, $\kappa_0 = -2Nl/(Ebh^2)$ represents the dimensionless limit of the elastic curvature of the section, and $\xi_0(\tau)$ is the dimensionless abscissa along the beam of the section where κ_0 is reached (see Figure A.2).

In the case of **a simply supported beam (Rev.2, second revision)**, solution u can be expressed in the form

$$u(\xi, \tau) = \sqrt{2} \sin(\pi\xi) \eta(\tau) = \sqrt{2} \sin(\pi\xi) U(\theta), \quad (28)$$

with

$$\theta_\tau = \omega. \quad (29)$$

The potential function $V(U)$ of the beam becomes [17]

$$V_{lin}(U) = \frac{\pi^4 U^2}{2} \quad (30)$$

for $|U| < \frac{\kappa_0}{\pi^2 \sqrt{2}}$, and

$$V_{nl}(U) = \pi^4 U^2(\theta) \xi_0 - \frac{\pi^3 U^2(\theta)}{2} \sin(2\pi\xi_0) + 6\sqrt{2}\pi\kappa_0 |U(\theta)| \cos(\pi\xi_0) + \frac{16}{\pi} \sqrt{\sqrt{2}\kappa_0^3 \pi^2 |U(\theta)|} E\left(\frac{\pi}{4}(1-2\xi_0), 2\right) - 3\kappa_0^2 \xi_0 + \frac{3}{2}\kappa_0^2, \quad (31)$$

for $|U| \geq \frac{\kappa_0}{\pi^2 \sqrt{2}}$, where we use the elliptical integral

$$E\left(\frac{\pi}{4}(1-2\xi_0), 2\right) = \int_0^{\frac{\pi}{4}(1-2\xi_0)} \frac{1}{\sqrt{1-2\sin^2(\zeta)}} d\zeta. \quad (32)$$

The abscissa ξ_0 , representing the boundary of the cracked region along the beam (see Figure A.2), can be deduced from the relation

$$|\kappa(\xi_0, \theta)| = \kappa_0, \quad (33)$$

so that

$$\xi_0(U) = \frac{1}{\pi} \arcsin\left(\frac{\kappa_0}{\pi^2 \sqrt{2} |U|}\right), \quad (34)$$

for $|U| \geq \frac{\kappa_0}{\pi^2\sqrt{2}}$ and

$$\xi_0(U) = \frac{1}{2} \quad (35)$$

for $|U| < \frac{\kappa_0}{\pi^2\sqrt{2}}$.

3.1. Free damped oscillations

Equations (19) and (20) can be arranged to address free damped oscillations by putting $\bar{p} = 0$, which yields

$$\frac{2\pi}{\omega} = \oint \frac{dU}{\sqrt{2(a - V(U))}}, \quad (36)$$

$$\frac{a'}{\omega^2} = -\frac{\zeta}{\pi} \oint \sqrt{2(a - V)} dU. \quad (37)$$

Equations (36) and (37) can be written, **in the linear elastic case**, for $|U| < \frac{\kappa_0}{\pi^2\sqrt{2}}$,

$$\frac{\pi}{\omega} = 2 \int_0^{R_1} \frac{dU}{\sqrt{2(a - V_{lin}(U))}}, \quad (38)$$

$$\frac{a'}{\omega^2} = -\frac{4\zeta}{\pi} \int_0^{R_1} \sqrt{2(a - V_{lin}(U))} dU, \quad (39)$$

with R_1 the positive root of the equation

$$\mathbf{a} - \mathbf{V}_{lin}(\mathbf{U}) = \mathbf{0}. \quad (\text{Rev1, editorial comment 4}) \quad (40)$$

In the masonry-like case, for $|U| \geq \frac{\kappa_0}{\pi^2\sqrt{2}}$, we have

$$\frac{\pi}{\omega} = 2 \int_0^{\frac{\kappa_0}{\pi^2\sqrt{2}}} \frac{dU}{\sqrt{2(a - V_{lin}(U))}} + 2 \int_{\frac{\kappa_0}{\pi^2\sqrt{2}}}^{R_2} \frac{dU}{\sqrt{2(a - V_{nl}(U))}}, \quad (41)$$

$$\frac{a'}{\omega^2} = -\frac{4\zeta}{\pi} \int_0^{\frac{\kappa_0}{\pi^2\sqrt{2}}} \sqrt{2(a - V_{lin}(U))} dU - \frac{4\zeta}{\pi} \int_{\frac{\kappa_0}{\pi^2\sqrt{2}}}^{R_2} \sqrt{2(a - V_{nl}(U))} dU, \quad (42)$$

with R_2 the positive root of the equation

$$a - V_{nl}(U) = 0. \quad (43)$$

Given the initial conditions a_0 and ω_0 , we can solve the motion equation in terms of the slowly varying functions energy a and frequency ω .

In order to apply this model, we choose a form of the periodic function $U(\theta)$ as well. This can be accomplished by using the Fourier expansion. Provided that no internal resonances act on the first mode, we can limit ourselves to the first term and put

$$U(\theta) = A \cos(\theta) = A(\tau) \cos\left(\int_0^\tau \omega(\tau) d\tau + \theta_0\right) = \eta(\tau), \quad (44)$$

where A is a non-negative number representing the maximum "amplitude" of the transversal motion of the beam. Considering that the energy a of the system is a slowly varying function of time, a relation between a and A can be easily found by calculating, with the help of (16) the integral of a on the period 2π of the oscillation

$$a(A) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2}\omega^2 U_\theta^2 + V(U)\right) d\theta, \quad (45)$$

with U given by (44).

If A_0 is the initial amplitude, we are in a position to determine the initial energy of the system a_0 by means of (45), as well as the initial frequency of the beam by means of (38) and (39) or (41) and (42), respectively, for the linear and nonlinear case. Then, the modulation equations can be solved and the evolution of a and ω determined. Figures A.3 and A.4 show the parameters a and ω vs. τ for different values of the damping ratio ζ . Figures A.5 and A.6 show a plot of the transverse displacement of the beam vs. time for different values of ζ , for a beam made of linear elastic (dashed line) and masonry-like (continuous line) material. Energy a decreases following an exponential law, as does the amplitude A of the motion. Instead, the frequency ω , starting at the initial value ω_0 , which is the same as in the absence of damping [17], tends to the linear elastic value π^2 . Depending on ζ , the values τ_{1el} and τ_{2el} represent the instants at which this linear elastic value is reached and the solution tends to assume the linear elastic form. At these times the beam will fall entirely in the linear elastic field and ξ_0 will towards $\frac{1}{2}$.

3.2. Forced damped oscillations

Equations (19) and (20) are now taken in their complete form. Before showing an application, some specifications are needed. Firstly, the unimodal assumption (28) involves some limitations on the form of the forcing term. In particular, a unimodal form of the solution is expected if the beam is

subjected to a sinusoidal forcing term whose frequency is close to the fundamental linear elastic one [30, 31]. This is precisely the case we will study, by considering a forcing term of the form

$$\bar{p} = \bar{k} \sin(\pi^2 \tau + \lambda \tau), \quad (46)$$

where

$$\bar{k} = \int_0^1 k(\xi) \phi(\xi) d\xi \quad (47)$$

and λ is small.

Secondly, the forcing terms in the modulation equations (19) and (20) cannot be dealt with without choosing an "a priori" form for $U(\theta)$. We know that the response over time of nonlinear elastic systems subjected to primary resonance tends to the forcing action frequency and exhibit a phase shift that can be considered a slowly varying function of time [30, 31]. Thus, a possible form for $U(\theta)$ is

$$U(\theta) = A \sin(\theta), \quad (48)$$

where

$$\theta = (\pi^2 + \lambda)\tau - \gamma(T), \quad (49)$$

Equation (46) thus becomes

$$\bar{p} = \bar{k} \sin(\theta + \gamma) \quad (50)$$

and, by introducing (48) and (50) into (19) and (20), the modulation equations take the form

$$\frac{\omega}{2\pi} \oint \frac{dU}{\sqrt{2(a - V(U))}} - 1 = -\frac{\bar{k}}{2} \frac{\partial A}{\partial a} \cos \gamma, \quad (51)$$

$$\frac{a'}{2\pi} \oint \frac{dU}{\sqrt{2(a - V(U))}} = \oint -\frac{\zeta \omega}{\pi} \sqrt{2(a - V(U))} dU + \frac{\bar{k}}{2} A \sin \gamma \quad (52)$$

and

$$\omega = \theta' = \pi^2 + \lambda - \gamma'. \quad (53)$$

As for free oscillations, we can find a relation between the amplitude A of the motion and the energy of the system a (45) by using expression (48) for U and the approximation $\omega \simeq \pi^2 + \lambda$ for the oscillation frequency. Expression (45) can then be inverted to obtain

the function $A(a)$ we needed to solve the modulation equations (51) and (52). (Rev 1, editorial comment 5).

The stationary values of the parameters can easily be deduced via Equations (51) and (52) by putting $a' = \gamma' = 0$. The system thereby reduces to

$$\frac{\pi^2 + \lambda}{2\pi} \oint \frac{dU}{\sqrt{2(a - V(U))}} - 1 = -\frac{\bar{k}}{2} \frac{\partial A}{\partial a} \cos \gamma, \quad (54)$$

$$\oint \frac{\zeta(\pi^2 + \lambda)}{\pi} \sqrt{2(a - V(U))} dU = \frac{\bar{k}}{2} A(a) \sin \gamma \quad (55)$$

and frequency ω tends towards its stationary value

$$\omega = \pi^2 + \lambda. \quad (56)$$

Remark. It seems worthwhile noting that in [15], [16] a different approach has been proposed, by which the assumed form of the displacement is introduced directly in the Lagrangian (5) and integrations performed explicitly. The Euler–Lagrange equations thus lead to the system

$$G(A) + \frac{A\pi}{2}(\lambda - \gamma') + \frac{\bar{k}}{\pi^2} \cos \gamma = 0, \quad (57)$$

$$\frac{A'\pi}{2} + \frac{\zeta\pi^2}{2} A - \frac{\bar{k}}{\pi^2} \sin \gamma = 0, \quad (58)$$

for the slowly varying parameters A and γ , and function $G(A)$ assumes the

explicit form

$$\begin{aligned}
G(A) &= \\
&= \frac{A\pi^3}{4} - \frac{A\pi^2}{2} \arcsin\left(\frac{\kappa_0}{A\pi^2}\right) + \frac{A\pi^2}{2} \frac{\kappa_0}{\sqrt{A^2\pi^4 - \kappa_0^2}} + \\
&- 2A\pi^4 \int_0^{\frac{1}{\pi^2} \arccos\left(\frac{\kappa_0}{A\pi^2}\right)} \cos^2(\pi^2\tau) \xi_0 d\tau - \frac{\kappa_0^3}{2A\pi^2} \frac{1}{\sqrt{A^2\pi^4 - \kappa_0^2}} + \\
&- 10\pi^3 \kappa_0 A \int_0^{\frac{1}{\pi^2} \arccos\left(\frac{\kappa_0}{A\pi^2}\right)} \frac{\cos^2(\pi^2\tau)}{\sqrt{A^2\pi^4 \cos^2(\pi^2\tau) - \kappa_0^2}} d\tau + \\
&+ 8\pi \kappa_0^2 \int_0^{\frac{1}{\pi^2} \arccos\left(\frac{\kappa_0}{A\pi^2}\right)} \sqrt{\frac{\cos(\pi^2\tau)}{\kappa_0 A}} \left(\int_{\xi_0}^{\frac{1}{2}} \sqrt{\sin \pi \xi} d\xi \right) d\tau + \\
&+ \frac{10\kappa_0^3}{A\pi} \int_0^{\frac{1}{\pi^2} \arccos\left(\frac{\kappa_0}{A\pi^2}\right)} \frac{1}{\sqrt{A^2\pi^4 \cos^2(\pi^2\tau) - \kappa_0^2}} d\tau.
\end{aligned} \tag{59}$$

The results obtained using (57) and (58) and (19) and (20) are in considerably good agreement. **The approach proposed in the present paper thus offers a more general form of the modulation equations and, at least in the case of free oscillations, allows solving system (19) and (20) without having to make any assumption on the explicit form of $U(\theta)$. (Rev. 1, comment 3.)**

4. Some example applications

Some numerical tests have been performed, using the scheme shown in Figure A.7. Three values were chosen for slenderness, with the corresponding section height h equal to 0.30 m, 0.40 m and 0.50 m; two values of damping ratio ζ were considered: 2% and 5%. The beam is subjected to a sinusoidal load of variable amplitude k and frequency $(\nu_e + \lambda)$. For the three slenderness values chosen, the numerical values of the fundamental frequency are $\nu_e(0.3 \text{ m}) = 4.9 \text{ Hz}$, $\nu_e(0.4 \text{ m}) = 6.5 \text{ Hz}$ and $\nu_e(0.5 \text{ m}) = 8.1 \text{ Hz}$. For all tests, null initial displacements and velocities have been imposed on the beam. Some tests have been performed for a beam made of a linear elastic material too.

The analytical results have been compared with those obtained via the NOSA-ITACA code (<https://www.nosaitaca.it>), in which the masonry-like constitutive equation has been implemented, for both static and dynamic

problems. *NOSA-ITACA is a general purpose finite element code, in which masonry materials are modelled by means of an explicit nonlinear relation between the Cauchy stress tensor and the total strain tensor [24]. (Rev.2, second revision)* With the aim of optimizing the comparisons between the analytical and numerical results, different kinds of elements have been tested, while varying the number of elements as well. Lastly, the eight-node isoparametric thin-shell element described in [24] was chosen and the beam divided into 120 finite-elements of equal length. (Rev.1, comment 4). *The good agreement between numerical and analytical results shown in Figures from A.8 to A.11 and from A.14 to A.17 allows us to assess the validity limits of the model. In particular, in all examples the contribution given by the coupling effect is negligible. In fact, the NOSA-ITACA approach to the dynamic problem is totally general and no assumptions are made on the uncoupling between the axial and the flexural behaviour of the structure. (Rev.2, second revision)* Some other comparisons can be found in [15] and [17], where the analytical results are tested via the MADY code *in which equation (26) is explicitly implemented. (Rev.2, second revision)*

Figures A.8 and A.9 show the displacements of the beam mid-point $v_{L/2}$ for $h = 0.40$ m, $k = 400$ N/m and different values of the damping ratio ζ . In both cases, the analytical and numerical results are quite consistent. Note that, after a brief transient stage, the oscillations tend toward stationary behaviour. Figures A.10 and A.11 instead show the stress σ_x at the extrados of the mid-section vs. t for $h = 0.40$ m, $k = 400$ N/m and different values of damping ratio ζ . Figures A.12 and A.13 show the behaviour of amplitude A and the phase displacement

$$\beta = \lambda t - \gamma \quad (60)$$

vs. t for $\lambda = 0$, $k = 400$ N/m, $h = 0.4$ m and different values of ζ . When compared with Figures A.8 and A.9, these figures confirm the slow variation of the parameters A and γ . Note that the curves in Figure A.13 start at the same value $\beta = -\frac{\pi}{2}$, corresponding to the linear elastic solution to the problem. Figures A.14 and A.15 show the stationary amplitude $\bar{A} = \sqrt{2}A$ vs. k for different values of h and ζ . The nonlinear values are quite different from the corresponding linear ones and all curves tend to exhibit very marked softening behaviour. It is worth noting that the curves related to the different damping values in Figure A.15 tend to coincide for large values of k . All

the numerical amplitude values are lightly lower than the corresponding analytical ones, that is to say: the numerical tests tend to reveal more softening behaviour. Figure A.16 shows the phase displacements β vs. k ; **the numerical values were obtained using the Fast Fourier Transform, while the horizontal line for $\beta = -\frac{\pi}{2}$ represents the linear elastic solution.** (Rev1, comment 5) Lastly, Figure A.17 shows a comparison between the linear elastic frequency response function (dashed curve), the corresponding analytical nonlinear function (continuous curve) and the results of the numerical tests with variable frequency excitations (red curve). The differences between the linear and nonlinear responses are considerable, particularly in the range centred on the linear fundamental frequency. The nonlinear analytical curve presents the typical shift towards low frequencies characteristic of softening systems. Moreover, for excitations of given frequency and amplitude, the curve presents more than one solution, depending on the initial conditions. In our tests, with the chosen initial conditions, the numerical solution presents a jump at about 5.6 Hz, from the upper to the lower branch of the analytical curve.

5. Conclusions

An analytical method has been presented to study the periodic oscillation of masonry beam–columns, under some hypotheses on the geometry and form of the solution. The nonlinear behaviour of masonry has been taken into account by means of a masonry–like constitutive equation expressed in terms of generalized stresses and strains. Some example applications have been shown and the analytical results compared to those obtained via the finite element code NOSA–ITACA. The analytical and numerical results have proven to be consistently in good agreement. The numerical methods enable solving problems for very general conditions of geometry and loading. However, the analytical solutions, albeit limited to some particular cases, provide concise descriptions of the nonlinear phenomena involved and contribute to a better understanding of the overall behaviour of masonry structures. **All the results presented herein were obtained for transverse vibrations only and assuming a unimodal expression for the beam’s displacements.** Future work is planned to consider *other kinds of dynamic loading for which the effects of higher modes and of the nonlinear interaction between longitudinal and transverse displacements can be no more negligible.* (Rev2, second revision)

Appendix A. Appendix

The averaged Lagrangian method has been described in section 2 and applied to masonry-like materials in sections 3 and 4. However, finding a formal justification for this powerful and intuitive method is not an easy matter. In [35] such a justification is furnished via the multiple scales method: Whitham applies the multiple scales or "two-timing" method to the study of nonlinear dispersive waves and demonstrates analogous results by writing the variational equations for the averaged Lagrangian. In this Appendix Whitham's scheme is followed to prove the effectiveness of the averaged Lagrangian method for nonlinear elastic beams subjected to small nonconservative actions.

Firstly, let us use the multiple scales method [30, 35] to find an approximate solution to (9). To this end, if ε is a small dimensionless parameter of the same order of magnitude as the amplitude of the beam's transverse displacement, we can introduce the new variable

$$T = \varepsilon\tau, \quad (\text{A.1})$$

which in same sense measures the "slow" time scale of the problem. We can explicitly split function η defined by (7) into its fast and slow oscillating parts and write

$$\eta = U(\theta, T; \varepsilon), \quad (\text{A.2})$$

with

$$\theta = \varepsilon^{-1}\Theta(T), \quad \theta_\tau = \omega. \quad (\text{A.3})$$

In (A.3) θ represents the phase of the periodic function η , while its derivative θ_τ with respect to time is the oscillation frequency ω . Note that ω is a slowly varying function:

$$\omega = \varepsilon^{-1}\Theta_\tau = \Theta_T. \quad (\text{A.4})$$

The time derivatives can now be scaled so that

$$\frac{\partial}{\partial\tau} = \omega \frac{\partial}{\partial\theta} + \varepsilon \frac{\partial}{\partial T}. \quad (\text{A.5})$$

Now, by putting

$$\bar{L}_1 = \frac{\partial \bar{L}}{\partial \eta_\tau}, \quad \bar{L}_2 = \frac{\partial \bar{L}}{\partial \eta}, \quad (\text{A.6})$$

equation (9) becomes

$$\omega \frac{\partial \bar{L}_1}{\partial \theta} + \varepsilon \frac{\partial \bar{L}_1}{\partial T} - \bar{L}_2 = -\varepsilon \left(2\bar{\mu}\omega \frac{\partial U}{\partial \theta} - \bar{p} \right), \quad (\text{A.7})$$

with \bar{L} given by (10), $\bar{\mu}\varepsilon = \mu$, $\bar{p}\varepsilon = \bar{p}$ and we have neglected the terms of the ε^2 order. In (A.7) the smallness of the nonconservative terms is expressed explicitly by means of ε .

We can write (A.7) in the form

$$\frac{\partial}{\partial \theta} (\omega \bar{L}_1 \frac{\partial U}{\partial \theta} - \bar{L}) + \varepsilon \frac{\partial}{\partial T} (\bar{L}_1 \frac{\partial U}{\partial \theta}) = -\varepsilon \left(2\bar{\mu}\omega \frac{\partial U}{\partial \theta} - \bar{p} \right) \frac{\partial U}{\partial \theta}. \quad (\text{A.8})$$

Indeed, with the help of (A.5), we have

$$\frac{\partial \bar{L}}{\partial \theta} = \bar{L}_1 \frac{\partial}{\partial \theta} (\omega \frac{\partial U}{\partial \theta} + \varepsilon \frac{\partial U}{\partial T}) + \bar{L}_2 \frac{\partial U}{\partial \theta} = \bar{L}_1 (\omega \frac{\partial^2 U}{\partial \theta^2} + \varepsilon \frac{\partial^2 U}{\partial \theta \partial T}) + \bar{L}_2 \frac{\partial U}{\partial \theta}. \quad (\text{A.9})$$

By introducing (A.9) into (A.8) and performing all the calculations we obtain

$$\frac{\partial U}{\partial \theta} \left(\omega \frac{\partial \bar{L}_1}{\partial \theta} + \varepsilon \frac{\partial \bar{L}_1}{\partial T} - \bar{L}_2 \right) = -\varepsilon \left(2\bar{\mu}\omega \frac{\partial U}{\partial \theta} - \bar{p} \right) \frac{\partial U}{\partial \theta} \quad (\text{A.10})$$

which is equivalent to (A.7).

By taking the expressions

$$U = \sum_{n=0}^{\infty} \varepsilon^n U^{(n)}, \quad (\text{A.11})$$

$$\bar{L} = \sum_{n=0}^{\infty} \varepsilon^n \bar{L}(U^{(n)}, \omega U_{\theta}^{(n)}) = \sum_{n=0}^{\infty} \varepsilon^n \bar{L}^{(n)}, \quad (\text{A.12})$$

equation (A.8) can be expanded into the set

$$\frac{\partial}{\partial \theta} (\omega \bar{L}_1^{(0)} \frac{\partial U^{(0)}}{\partial \theta} - \bar{L}^{(0)}) = 0, \quad (\text{A.13})$$

$$\frac{\partial}{\partial \theta} (\omega \bar{L}_1^{(1)} \frac{\partial U^{(1)}}{\partial \theta} - \bar{L}^{(1)}) = -\frac{\partial}{\partial T} (\bar{L}_1^{(0)} \frac{\partial U^{(0)}}{\partial \theta}) - 2\mu\omega \left(\frac{\partial U^{(0)}}{\partial \theta} \right)^2 + \bar{p} \frac{\partial U^{(0)}}{\partial \theta}, \quad (\text{A.14})$$

where, again, we have neglected terms of the ε^2 order.

Equation (A.13) yields

$$\omega \bar{L}_1^{(0)} \frac{\partial U^{(0)}}{\partial \theta} - \bar{L}^{(0)} = a(T), \quad (\text{A.15})$$

where $a(T)$ is a slowly varying parameter related to the energy of the system. By virtue of (14), equation (A.15) can be expressed as

$$\frac{1}{2} \omega^2 \left(\frac{\partial U^{(0)}}{\partial \theta} \right)^2 + V(U^{(0)}) = a(T). \quad (\text{A.16})$$

Remembering that U , U_θ are periodic functions with period 2π , the secular condition on U can be imposed on (A.14) to arrive at

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial T} (\bar{L}_1^{(0)} \frac{\partial U^{(0)}}{\partial \theta}) d\theta = \mathcal{D} + \mathcal{P}, \quad (\text{A.17})$$

with

$$\mathcal{D} = \frac{1}{2\pi} \int_0^{2\pi} -2\mu\omega \left(\frac{\partial U^{(0)}}{\partial \theta} \right)^2 d\theta = \oint -\frac{\zeta\omega}{\pi} \sqrt{2(a-V)} dU^{(0)} \quad (\text{A.18})$$

and

$$\mathcal{P} = \frac{1}{2\pi} \int_0^{2\pi} \bar{p} dU^{(0)}. \quad (\text{A.19})$$

Equations (A.13) and (A.17) furnish the approximate solution we are looking for: equation (A.13) provides the form of the solution, while (A.17) gives us the modulation.

Let us now consider the variational principle

$$\delta \int \frac{1}{2\pi} \int_0^{2\pi} \bar{L}(U, \omega \frac{\partial U}{\partial \theta} + \varepsilon \frac{\partial U}{\partial T}) d\theta dT = 0 \quad (\text{A.20})$$

where the assumptions (A.2) and (A.3) have been introduced in the Lagrangian (13). The explicit separation of θ from T provided by (A.2) allows us to consider a and ω constant with respect to θ , when performing integration.

In absence of damping and forcing terms variations δU give [14], [35]

$$\omega \frac{\partial}{\partial \theta} \bar{L}_1 + \varepsilon \frac{\partial}{\partial T} \bar{L}_1 - \bar{L}_2 = 0, \quad (\text{A.21})$$

while variations $\delta\Theta$ (involved through $\omega = \Theta_T$) furnish

$$\frac{\partial^2 \mathcal{L}}{\partial T \partial \omega} = 0 \quad (\text{A.22})$$

with \mathcal{L} the averaged Lagrangian given by (16).

We can generalize these results by the extended Hamilton principle [29], [30]. Equation (A.21) becomes then

$$\omega \frac{\partial}{\partial \theta} \bar{L}_1 + \varepsilon \frac{\partial}{\partial T} \bar{L}_1 - \bar{L}_2 = -\varepsilon \left(2\bar{\mu}\omega \frac{\partial U}{\partial \theta} - \bar{p} \right) \quad (\text{A.23})$$

whose right member represents the component along U of the vector of the generalized forces (damping and external forces) acting upon the system. Equation (A.21) is equivalent to equation (A.7), which we have obtained by direct manipulation of the motion equation. Instead, equation (A.22) becomes

$$\frac{\partial^2 \mathcal{L}}{\partial T \partial \omega} = -\frac{1}{2\pi} \int_0^{2\pi} 2\bar{\mu}\omega \left(\frac{\partial U}{\partial \theta} \right)^2 d\theta + \frac{1}{2\pi} \int_0^{2\pi} \bar{p} \frac{\partial U}{\partial \theta} d\theta \quad (\text{A.24})$$

where this time the right member is the component along $\Theta = \varepsilon \theta$ of the vector of the averaged generalized forces. By introducing expressions (A.11) and (A.12) into (A.21) and (A.22) and neglecting terms of order higher than ε^2 we obtain

$$\frac{\partial}{\partial \theta} (\omega \bar{L}_1^{(0)} \frac{\partial U^{(0)}}{\partial \theta} - \bar{L}^{(0)}) = 0, \quad (\text{A.25})$$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial T} (\bar{L}_1^{(0)} \frac{\partial U^{(0)}}{\partial \theta}) d\theta = \mathcal{D} + \mathcal{P}. \quad (\text{A.26})$$

Equations (A.25) and (A.26) are indeed the same as (A.13) and (A.17).

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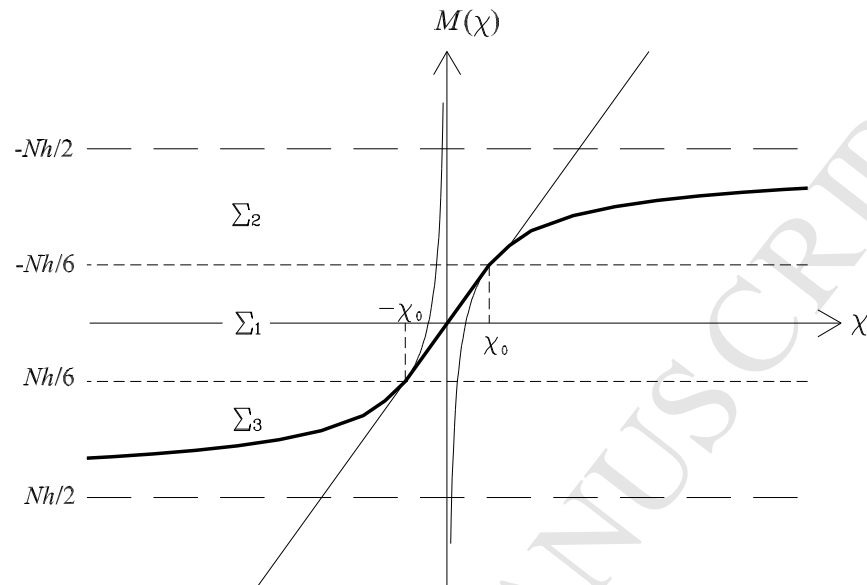


Figure A.1: The constitutive equation $M - \chi$ for a rectangular section made of a masonry-like material with zero tensile strength and infinite compressive strength.

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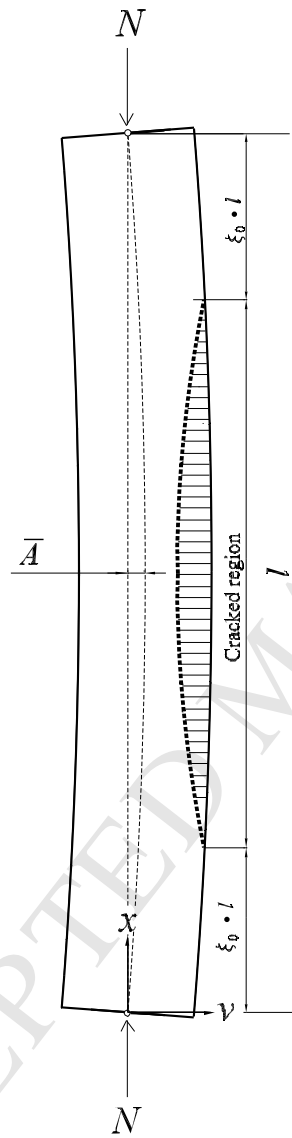


Figure A.2: A masonry beam-column

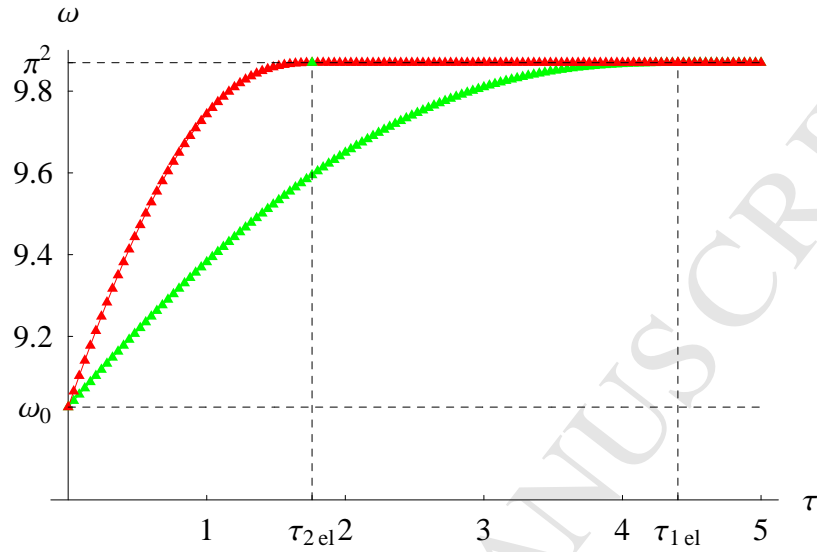


Figure A.3: Frequency ω vs. τ for $\zeta_1 = 0.02$ (green) and $\zeta_2 = 0.05$ (red).

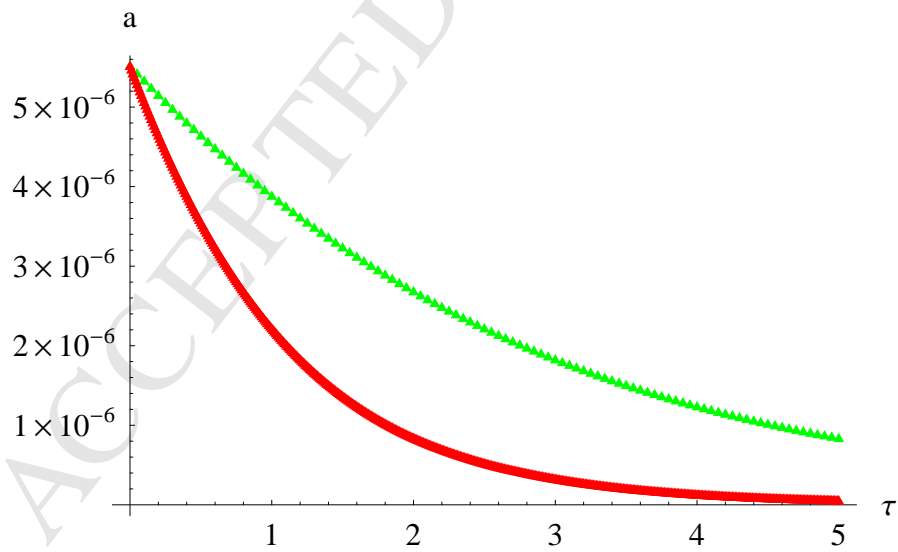


Figure A.4: Energy a vs. τ for $\zeta_1 = 0.02$ (green) and $\zeta_2 = 0.05$ (red).

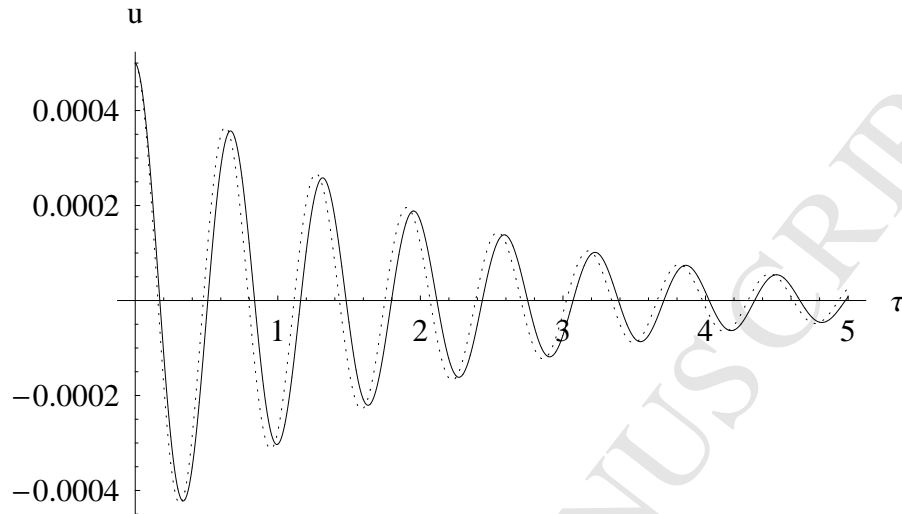


Figure A.5: Transverse displacements u of the beam midpoint vs. τ for $\zeta = 0.05$, $\kappa_0 = 0.002$ and $A_0 = 0.0005/\sqrt{2}$ in the linear (dotted) and nonlinear (continuous) case.

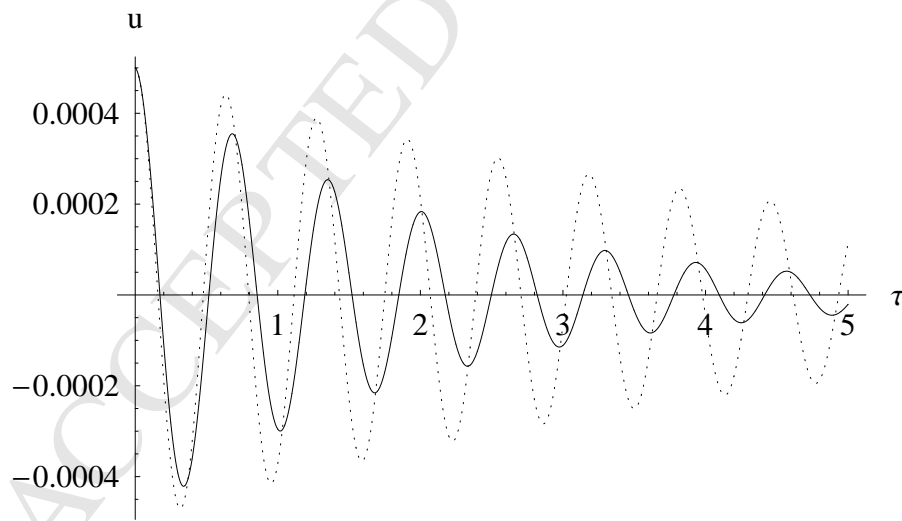


Figure A.6: Transverse displacements u of the beam midpoint vs. τ for $\zeta = 0.02$, $\kappa_0 = 0.002$ and $A_0 = 0.0005/\sqrt{2}$ in the linear (dotted) and nonlinear (continuous) case.

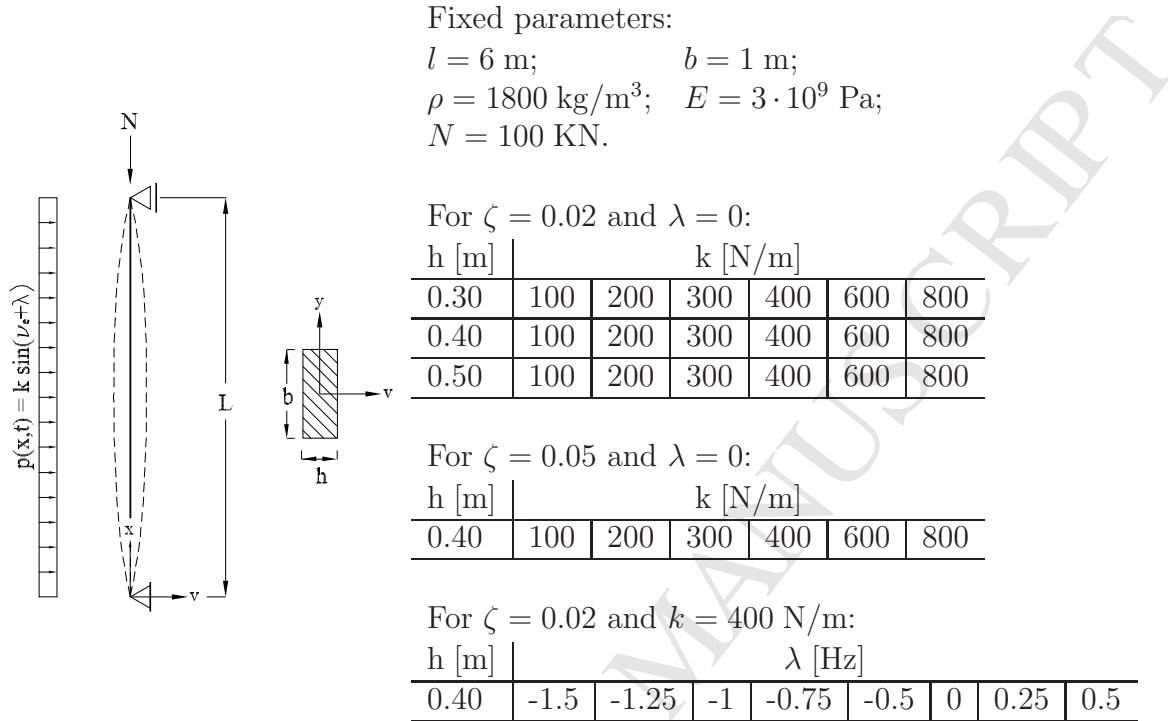
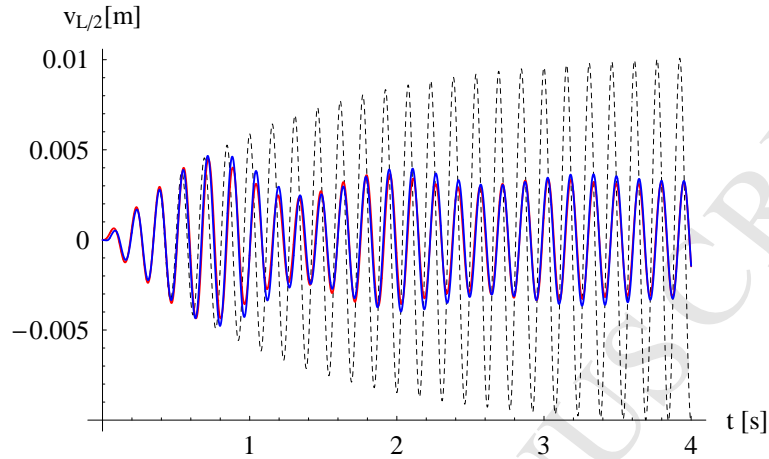


Figure A.7: Geometry of the beam and data used for the numerical tests.

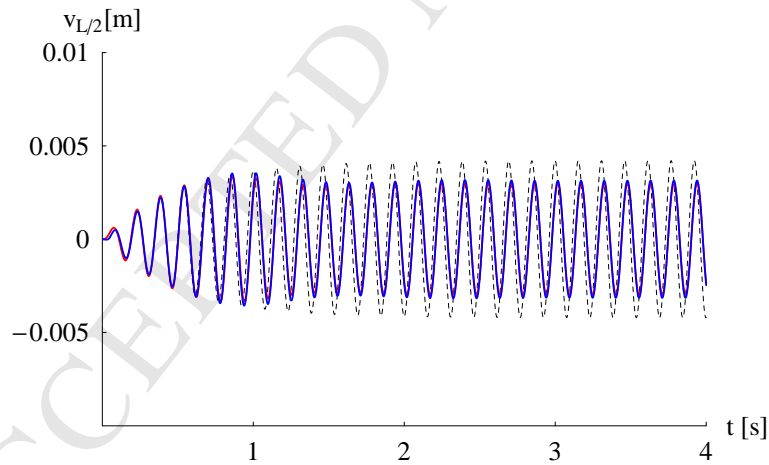
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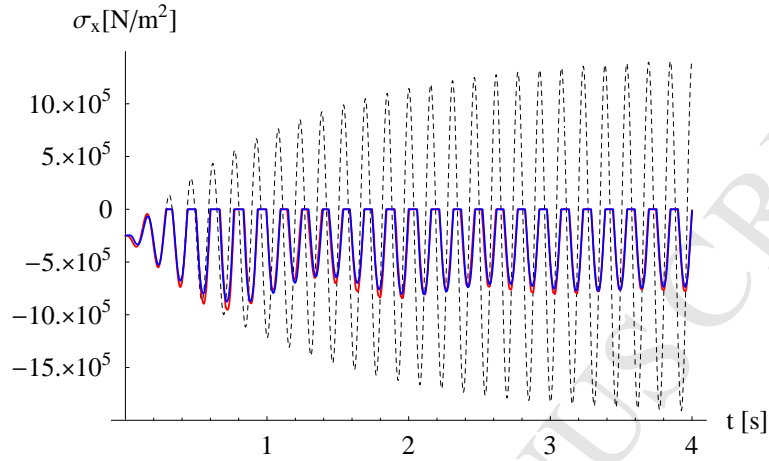
— nonlinear, numeric — nonlinear, analytic - - - linear elastic

Figure A.8: Displacements of the beam mid-section vs. time t for $\zeta = 0.02$, $k = 400$ N/m, $h = 0.4$ m.



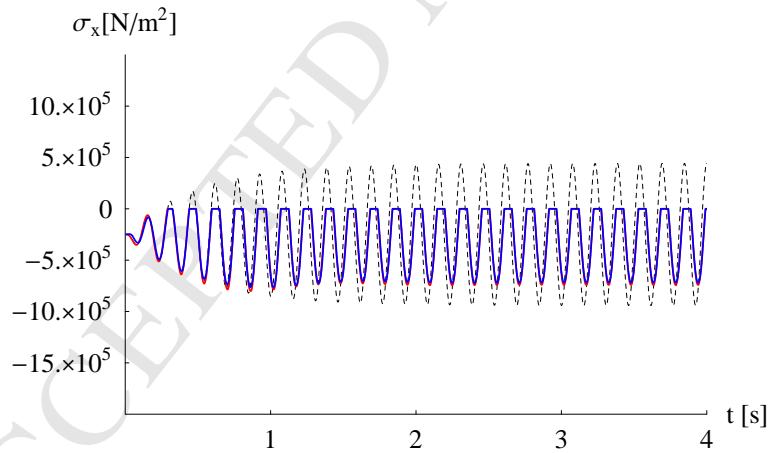
— nonlinear, numeric — nonlinear, analytic - - - linear elastic

Figure A.9: Displacements of the beam mid-section vs. time t for $\zeta = 0.05$, $k = 400$ N/m, $h = 0.4$ m.



— nonlinear, numeric — nonlinear, analytic - - - linear elastic

Figure A.10: Stress σ_x at the extrados of the beam mid-section vs. time t for $\zeta = 0.02$, $k = 400$ N/m, $h = 0.4$ m.



— nonlinear, numeric — nonlinear, analytic - - - linear elastic

Figure A.11: Stress σ_x at the extrados of the beam mid-section vs. time t for $\zeta = 0.05$, $k = 400$ N/m, $h = 0.4$ m.

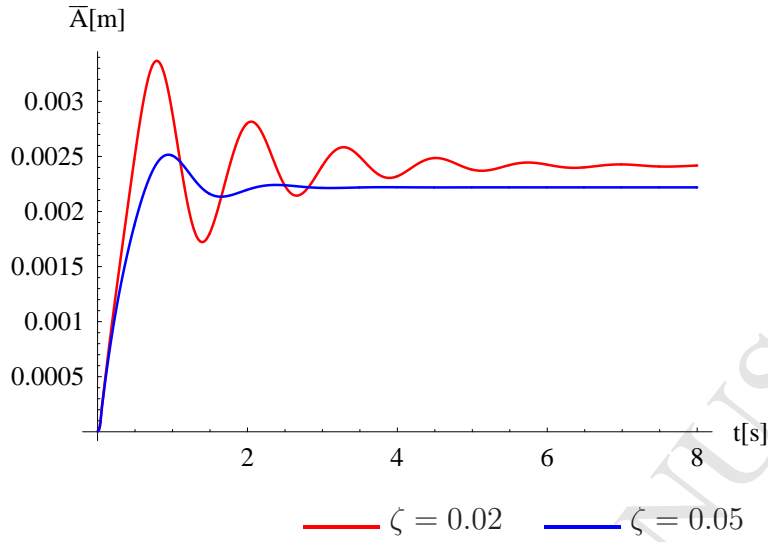


Figure A.12: Amplitude \bar{A} of displacements vs. time t for $k = 400$ N/m, $h = 0.4$ m, $\lambda = 0$.

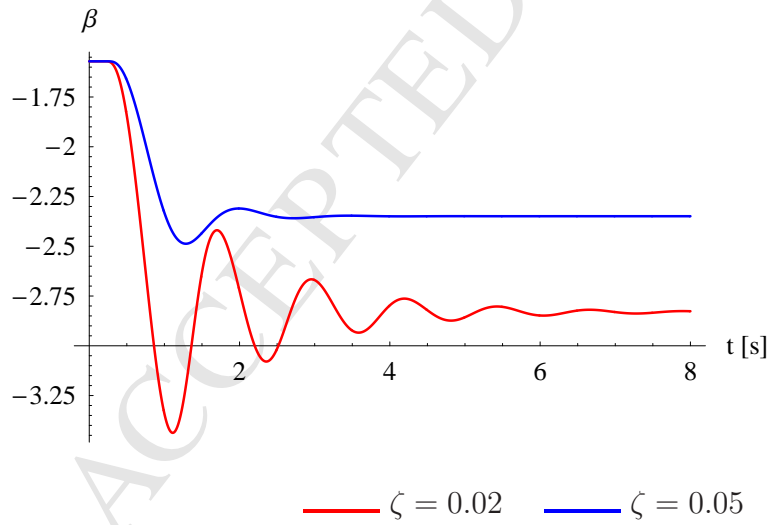


Figure A.13: Phase-shift β of displacements vs. time t for $k = 400$ N/m, $h = 0.4$ m, $\lambda = 0$.

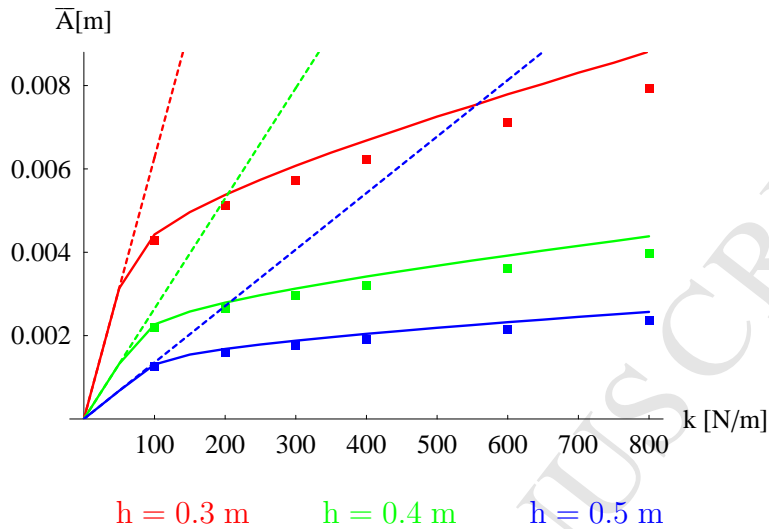


Figure A.14: Maximum stationary displacement of the beam vs. k for $\lambda = 0$, $\zeta = 0.02$ and different values of h . ■ nonlinear, numeric — nonlinear, analytic - - - linear elastic.

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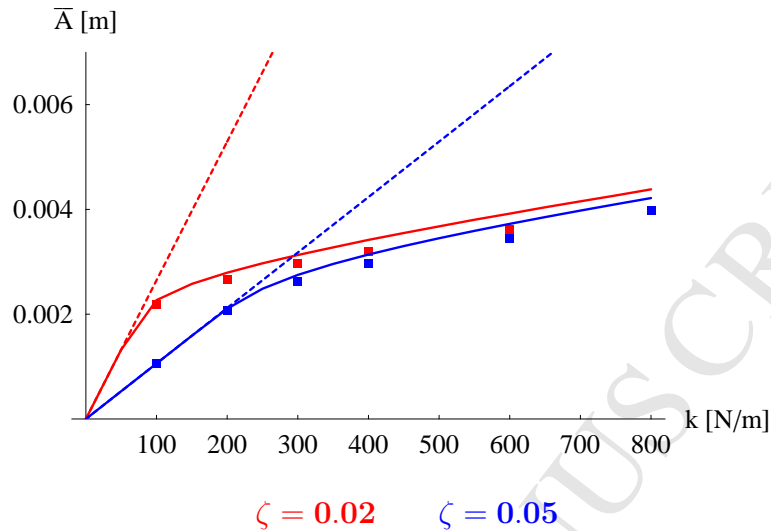


Figure A.15: Maximum stationary displacement of the beam vs. k for $\lambda = 0$, $h = 0.4$ m and different values of ζ . ■ nonlinear, numeric — nonlinear, analytic - - - - linear elastic.

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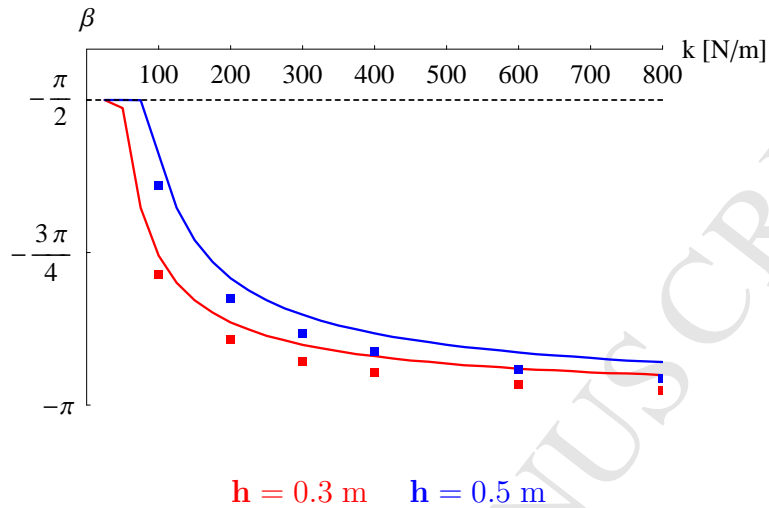
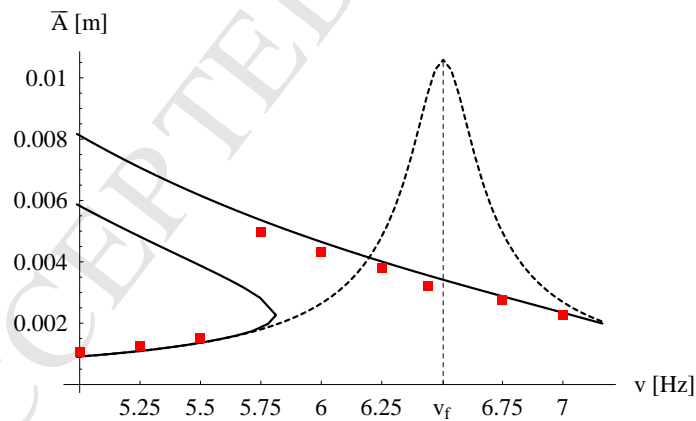


Figure A.16: Stationary values of β vs. k for $\lambda = 0$, $\zeta = 0.02$ and different values of h .

■ nonlinear, numeric — nonlinear, analytic - - - linear elastic. (Rev1, comment 5)



■ nonlinear, numeric — nonlinear, analytic - - - linear elastic

Figure A.17: Maximum stationary displacement of the beam vs. the excitation frequency ν for $\zeta = 0.02$ and $k = 400 \text{ N/m}$, $h = 0.4 \text{ m}$.

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