On the dynamic behaviour of masonry beam–columns: an analytical approach

Maria Girardi*[,a]

[a]*Istituto di Scienza e Tecnologie dell'Informazione "A. Faedo", ISTI–CNR*
Via G. Moruzzi 1, 56124 Pisa, Italy

Abstract

The paper presents an analytical approach to the study of the transverse vibrations of masonry beam–columns. Starting with the constitutive equation for beams made of a masonry–like material and the averaged Lagrangian of the system, some explicit approximate solutions are found to the problem of free damped periodic oscillations and forced oscillations in the case of primary resonance on the beam's first mode. In particular, a set of equations is obtained that gives the modulation over time of the system's energy and of the fundamental frequency of the beam's response. The analytical results are compared to those obtained via the finite element code NOSA–ITACA, developed at ISTI–CNR.

Key words: masonry–like materials, nonlinear dynamics, averaged Lagrangian method

1. Introduction

A constitutive model is proposed in [3], [20] for masonry–like materials with zero tensile strength and infinite compressive strength, where the constitutive equation for masonry–like materials [4], [5], [12], is specialized for masonry beams. The nonlinear elastic equation provided in [3], [20], which expresses the internal forces, normal force and bending moment, as functions of the generalized strains, stretching and change of curvature of the beam axis, has proven to be simple enough to enable some explicit calculations [3],

*Corresponding Author, Phone: +39 0503153075, Fax: +39 0503152040
Email address: Maria.Girardi@isti.cnr.it (Maria Girardi)

[8], [9], [20]. At the same time, its numerical implementation in the MADY code [10], [13], [14], represents a quick and effective way to asses the effects of the load's eccentricity on the static and dynamic behaviour of masonry columns, arches and towers. For cyclic actions, this approach can furnish reasonable results for slender structures, for which the influence of shear forces on the dynamic equilibrium tends to decrease and the nonlinear behaviour is due essentially to the opening of cracks.

In [9] the authors present an analytical study of the transverse vibrations of masonry beam–columns based on the constitutive equation described in [3], [20]. They limit themselves to considering free vibrations and obtain an explicit relation between the fundamental frequency of the beam and amplitude of the displacement. In the present paper the study is generalized in order to include damped and forced oscillations. In order to simplify computations, use is made of the averaged Lagrangian method proposed by G.B. Whitham to study the modulation of nonlinear dispersive waves [2], [18], [19]. This method reduces the problem to the study of a set of nonlinear differential equations – the so–called modulation equations – for some parameters of the problem, specifically energy and frequencies, which, if the nonconservative terms are small, can be considered slowly varying over time. The averaged Lagrangian method, whose use in the present context is justified in the Appendix, allows obtaining the modulation equations without the manipulations typical of other conventional methods based on series development, such as the multiple scales method [16], [17].

The paper is divided into three parts. In the first, the averaged Lagrangian method is presented and the modulation equations obtained for a broad class of nonlinear elastic materials. In the second, the method is applied to masonry–like materials, in the case of free damped and forced damped oscillations. Finally, the third part presents a parametric study, by varying on the one hand the slenderness and modal damping coefficient of the structure and, on the other, the forcing amplitude and frequency. All results are compared with those obtained numerically via the finite element code NOSA–ITACA [12], http://www.nosaitaca.it/, developed at ISTI–CNR for static and dynamic equilibrium problems of masonry structures and constructions.

2. The averaged Lagrangian method

Let us consider a rectilinear beam with length l and rectangular cross section with height h and width b, subjected to a uniform axial force N and a transverse load per unit length q. The beam is made of a nonlinear elastic material described by constitutive equation $M(\chi)$, where χ is the curvature of the beam and M the bending moment. Function $M(\chi)$ is assumed to be continuously differentiable and its second derivative piecewise continuous. Let us denote by E and ρ the Young's modulus and the density of the material, respectively, and by $J = bh^3/12$ the moment of inertia of the beam's section. Let the dissipative forces be modeled by a small viscous damping term in the form $\bar{C}v_t$, where v_t is the time derivative of the transverse displacement v, and $\bar{C}$ is a constant (see Figure 2).

In order to work with dimensionless quantities, if x and t are, respectively, the abscissa along the beam's axis and the time, we define

$$\xi = \frac{x}{l}, \qquad \tau = \frac{t}{T_c}, \qquad u = \frac{v}{l}, \qquad \kappa = \chi l, \qquad p = \frac{q\,{T_c}^2}{\rho bhl}, \qquad C = \frac{\bar{C}T_c}{\rho bh} \tag{1}$$

with $T_c = l^2/c$ and $c = \sqrt{EJ/(\rho bh)}$ the elastic constant of the beam.

In the following we denote partial derivatives by both the compact notation, using subscripts, and the extended notation, using quotients, while primes denote total differentiation.

We assume the effects of both the shear strain and the rotary inertia to be negligible. Moreover, we limit ourselves to considering situations in which the flexural displacement $u(\xi, \tau)$ and its derivative $u_\xi(\xi, \tau)$ are small, so that we can neglect the effects of the axial force on the dynamic equilibrium of the beam and write

$$\kappa(\xi, \tau) = -u_{\xi\xi}(\xi, \tau). \tag{2}$$

In order to obtain small flexural displacements, we also consider the forcing term $p(\xi, \tau)$ to be small.

Under these hypotheses, putting

$$f(\kappa) = \frac{l}{EJ}M(\kappa/l), \tag{3}$$

the equation of motion is

$$\frac{\partial^2 u}{\partial \tau^2} - \frac{\partial^2 (f \circ \kappa)}{\partial \xi^2} = p - C\,\frac{\partial u}{\partial \tau}. \tag{4}$$

Now, let

$$L\big(u(\tau,\xi)\big) = \frac{1}{2}(u_\tau)^2 - F(-u_{\xi\xi}) \tag{5}$$

be the Lagrangian, where F is the primitive of f such that $F(0) = 0$. Thus, equation (4) can be written in the form

$$\frac{\partial}{\partial\tau}\left(\frac{\partial L}{\partial u_\tau}\right) - \frac{\partial^2}{\partial\xi^2}\left(\frac{\partial L}{\partial u_{\xi\xi}}\right) = p - C\,u_\tau. \tag{6}$$

We are looking for an approximate solutions to (6) of the form [2, 16, 17]

$$u(\xi,\tau) = \phi(\xi)\,\eta(\tau), \tag{7}$$

where $\phi(\xi)$ is a periodic function of the span of the beam, with

$$\int_0^1 \phi^2 d\xi = 1, \tag{8}$$

and $\eta(\tau) = U(\theta)$ is a periodic function with period 2π, phase θ and frequency $\omega = \theta_\tau$. In addition, U_θ is a periodic function with period 2π as well. It is a simple matter to verify [1], [9], that η satisfies the equation

$$\frac{\partial}{\partial\tau}\frac{\partial\bar{L}}{\partial\eta'} - \frac{\partial\bar{L}}{\partial\eta} = -2\mu\eta' + \bar{p} \tag{9}$$

where

$$\bar{L} = \int_0^1 L(\phi,\phi'',\eta,\eta')d\xi, \tag{10}$$

$$\mu = \int_0^1 C\phi^2\,d\xi = \zeta\omega, \tag{11}$$

with ζ the damping ratio calculated under the hypothesis of modal damping [1], and

$$\bar{p} = \int_0^1 p\,\phi\,d\xi. \tag{12}$$

These hypotheses may be viewed as a rather strong restriction on the generality of the problem. However, provided that no internal resonance effects [16] occur, the unimodal expression (7) has proven to work quite well in a large number of cases, such as the study of the primary resonance on the first mode. In addition, its simplicity enables tackling the difficult calculations typically involved in nonlinear dynamical problems.

We now introduce the averaged Lagrangian [2],[18], [19]

$$\mathscr{L} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 L\ d\xi d\theta = \frac{1}{2\pi}\int_0^{2\pi} \bar{L}(U, \omega U_\theta)\, d\theta \tag{13}$$

where

$$\bar{L} = \bar{L}(U, \omega U_\theta) = \frac{1}{2}\omega^2 U_\theta^2 - V(U) \tag{14}$$

and

$$V(U) = \int_0^1 F(-\phi''\eta) d\xi \tag{15}$$

is a potential function.

For conservative systems, when $\mu = \bar{p} = 0$, a first integral of the equation of the motion (9) can be found in the form (see Appendix)

$$\frac{1}{2}\omega^2 U_\theta^2 + V(U) = a \tag{16}$$

where a is a constant representing the total energy of the system. Actually, provided that the nonconservative terms are sufficiently small, equation (16) can work for nonconservative systems as well. In this case a is no longer a constant but a slowly varying function of time, approximately constant during each loop of the oscillations. The same holds for frequency ω, which can be regarded as the "slow" variation of the phase θ over the time (see the Appendix). With the help of (14) and (16), the averaged Lagrangian (13) becomes,

$$\mathscr{L} = \frac{1}{2\pi}\int_0^{2\pi} (\frac{1}{2}\omega^2 U_\theta^2 - V(U)) d\theta = \frac{\omega^2}{2\pi}\int_0^{2\pi} U_\theta^2 d\theta - a = \tag{17}$$

$$= \frac{\omega}{2\pi}\oint \sqrt{2(a-V)} dU - a = \mathscr{L}(a, \omega) \tag{18}$$

where integration with respect to θ and U is performed by assuming a and ω to be constant [19, 18]. The averaged Lagrangian $\mathscr{L}$ depends only on the parameters a and ω, whose variation over time – generally referred to as "modulation"– is given by the Euler–Lagrange equations for $\mathscr{L}$:

$$\frac{\omega}{2\pi}\oint\frac{dU}{\sqrt{2(a-V(U)}}-1=-\frac{1}{2\pi}\int_0^{2\pi}\bar{p}\,U_a\,d\theta \tag{19}$$

$$\frac{a'}{2\pi}\oint\frac{dU}{\sqrt{2(a-V(U)}}=\mathscr{D}+\mathscr{P} \tag{20}$$

with

$$\mathscr{D}=\frac{1}{2\pi}\int_0^{2\pi}-2\mu\omega\,U_\theta^2\,d\theta=-\oint\frac{\zeta\omega}{\pi}\sqrt{2(a-V)}\,dU \tag{21}$$

and

$$\mathscr{P}=\frac{1}{2\pi}\int_0^{2\pi}\bar{p}\,dU. \tag{22}$$

The righthand members of (19) and (20) represent the components of the vector of the generalized averaged forces acting upon the beam [15, 17]. Note that, for $\bar{p}=0$, equation (19) is analogous to that obtained in [9] for free undamped oscillations by directly manipulating the equation of motion, while (20) is an energy balance equation, where the damping dissipation and the energy injection by the forcing term are taken into account. Provided that a potential $V(U)$ is known, the equations (19) and (20) work for a broad class of nonlinear elastic materials.

3. Application to masonry–like beams

Let us briefly recall the constitutive equation for masonry–like material with zero tensile strength and infinite compressive strength proposed in [3] and [20].

Let χ be the curvature of the beam, M and N the generalized stress, bending moment and normal force, acting on the beam's section. If N is a known quantity along the beam, we can obtain a relation $M=M(\chi,N)$ between the bending moment and the curvature, directly. Thus, if we define

$$\chi_0=-\frac{2N}{Ebh^2}, \tag{23}$$

where χ_0 is the curvature corresponding to the elastic limit, when a triangular stress distribution is reached on the section, the constitutive equation becomes

$$\frac{M(\chi)}{\rho bh}=\begin{cases} c^2\chi & \text{for } |\chi|\le\chi_0,\\ c^2\chi_0\mathrm{Sign}(\chi)(3-2\sqrt{\frac{\chi_0}{|\chi|}}) & \text{for } |\chi|>\chi_0,\end{cases} \tag{24}$$

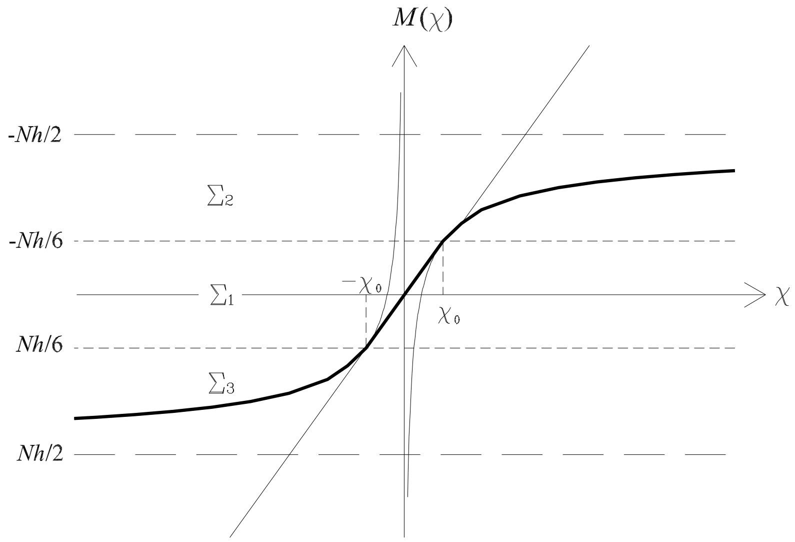


Figure 1: The constitutive equation $M-\chi$ for a rectangular section made of a masonry–like material with zero tensile strength and infinite compressive strength.

whose representation is given in Figure 1.

The constitutive equation (24) assumes dimensionless form by putting

$$F(\kappa)=\begin{cases}\frac{1}{2}\kappa^2 & \text{for} \quad \xi\le\xi_0,\\ \kappa_0\left(3\,|\kappa|-4\sqrt{\kappa_0\,|\kappa|}\right)+\frac{3}{2}\kappa_0^2 & \text{for} \quad \xi>\xi_0,\end{cases} \tag{25}$$

where F is the primitive of $f=Ml/(EJ)$ such that $F(0)=0$, $\kappa_0=-2Nl/(Ebh^2)$ represents the dimensionless limit of the elastic curvature of the section, and $\xi_0(\tau)$ is the dimensionless abscissa along the beam of the section where κ_0 is reached (see Figure 2).

In the case of beam hinged at its ends, the solution u can be expressed in the form

$$u(\xi,\tau)=\sqrt{2}\sin(\pi\xi)\,\eta(\tau)=\sqrt{2}\sin(\pi\xi)\,U(\theta), \tag{26}$$

with

$$\theta_\tau=\omega. \tag{27}$$

The potential function $V(U)$ of the beam becomes [9]

$$V_{lin}(U)=\frac{\pi^4U^2}{2} \tag{28}$$

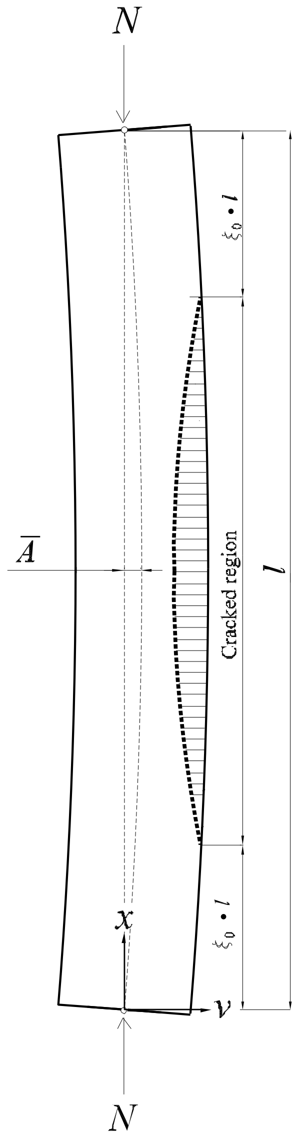


Figure 2: A masonry beam–column

for $|U| < \frac{\kappa_0}{\pi^2\sqrt{2}}$, and

$$V_{nl}(U) = \pi^4 U^2(\theta)\xi_0 - \frac{\pi^3 U^2(\theta)}{2}\sin(2\pi\xi_0) + 6\sqrt{2}\pi\kappa_0\,|\,U(\theta)|\cos(\pi\xi_0) + \\ -\frac{16}{\pi}\sqrt{\sqrt{2}\kappa_0^3\pi^2\,|\,U(\theta)|}\;E(\frac{\pi}{4}\,(1-2\xi_0),2) - 3\kappa_0^2\xi_0 + \frac{3}{2}\kappa_0^2, \tag{29}$$

for $|U| \geq \frac{\kappa_0}{\pi^2\sqrt{2}}$, where we use the elliptical integral

$$E(\frac{\pi}{4}\,(1-2\xi_0),2) = \int_0^{\frac{\pi}{4}(1-2\xi_0)} \frac{1}{\sqrt{1-2\sin^2(\zeta)}}\;d\zeta. \tag{30}$$

The abscissa ξ_0, representing the boundary of the cracked region along the beam (see Figure 2), can be deduced from the relation

$$|\kappa(\xi_0,\theta)| = \kappa_0, \tag{31}$$

so that

$$\xi_0(U) = \frac{1}{\pi}\arcsin\left(\frac{\kappa_0}{\pi^2\sqrt{2}\,|U|}\right), \tag{32}$$

for $|U| \geq \frac{\kappa_0}{\pi^2\sqrt{2}}$ and

$$\xi_0(U) = \frac{1}{2} \tag{33}$$

for $|U| < \frac{\kappa_0}{\pi^2\sqrt{2}}$.

3.1. Free damped oscillations

Equations (19) and (20) can be arranged to address free damped oscillations by putting $\bar{p} = 0$, which yields

$$\frac{2\pi}{\omega} = \oint \frac{dU}{\sqrt{2(a - V(U))}} \tag{34}$$

$$\frac{a'}{\omega^2} = -\frac{\zeta}{\pi}\oint \sqrt{2(a-V)}\,dU. \tag{35}$$

Equations (34) and (35) can be written, for a beam made of linear elastic material, for $|U| < \frac{\kappa_0}{\pi^2\sqrt{2}}$,

$$\frac{\pi}{\omega} = 2\int_0^{R_1} \frac{dU}{\sqrt{2(a - V_{lin}(U)}} \tag{36}$$

$$\frac{a'}{\omega^2} = -\frac{4\zeta}{\pi}\int_0^{R_1} \sqrt{2(a - V_{lin}(U))}\,dU \tag{37}$$

with R_1 the positive root of the equation

$$a - V_l(U) = 0. \tag{38}$$

In the masonry–like case, for $|U| \geq \frac{\kappa_0}{\pi^2\sqrt{2}}$, we have

$$\frac{\pi}{\omega} = 2\int_0^{\frac{\kappa_0}{\pi^2\sqrt{2}}} \frac{dU}{\sqrt{2(a - V_{lin}(U)}} + 2\int_{\frac{\kappa_0}{\pi^2\sqrt{2}}}^{R_2} \frac{dU}{\sqrt{2(a - V_{nl}(U)}} \tag{39}$$

$$\frac{a'}{\omega^2} = -\frac{4\zeta}{\pi}\int_0^{\frac{\kappa_0}{\pi^2\sqrt{2}}} \sqrt{2(a - V_{lin}(U))}\, dU - \frac{4\zeta}{\pi}\int_{\frac{\kappa_0}{\pi^2\sqrt{2}}}^{R_2} \sqrt{2(a - V_{nl}(U))}\, dU \tag{40}$$

with R_2 the positive root of the equation

$$a - V_{nl}(U) = 0. \tag{41}$$

Given the initial conditions a_0 and ω_0, we can solve the motion equation in terms of the slowly varying functions energy a and frequency ω.

In order to apply this model, we choose a form of the periodic function $U(\theta)$ as well. This can be accomplished by using the Fourier expansion. Provided that no internal resonances act on the first mode, we can limit ourselves to the first term and put

$$U(\theta) = A\cos(\theta) = A(\tau)\, \cos\left(\int_0^{\tau} \omega(\tau)d\tau + \theta_0\right) = \eta(\tau), \tag{42}$$

where A is a non–negative number representing the maximum "amplitude" of the transversal motion of the beam. Considering that the energy a of the system is a slowly varying function of time, a relation between a and A can be easily found by calculating, with the help of (16) the integral of a on the period 2π of the oscillation

$$a(A) = \frac{1}{2\pi}\int_0^{2\pi} \big(\frac{1}{2}\omega^2 {U_\theta}^2 + V(U)\big)d\theta, \tag{43}$$

with U given by (42).

If A_0 is the initial amplitude, we are in a position to determine the initial energy of the system a_0 by means of (43), as well as the initial frequency of the beam by means of (36) and (37) or (39) and (40), respectively, for the linear and nonlinear case. Then, the modulation equations can be solved and

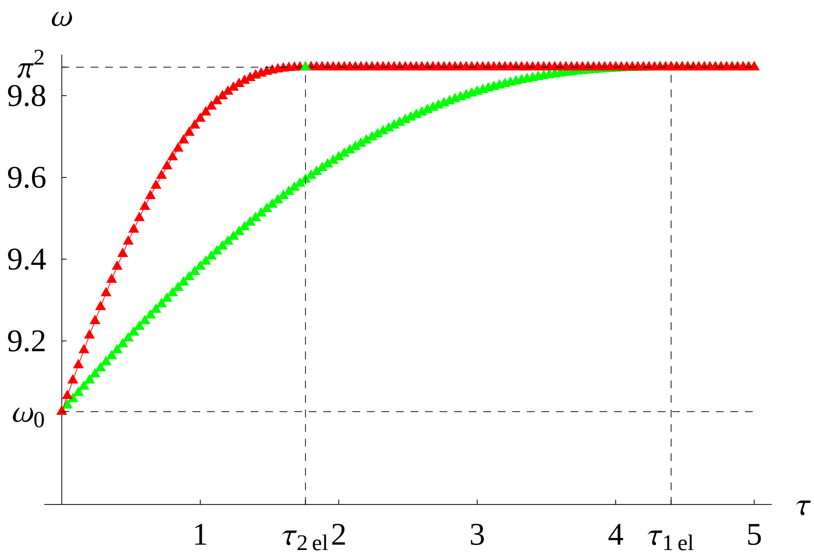


Figure 3: Frequency ω vs. τ for $\zeta_1 = 0.02$ (green) and $\zeta_2 = 0.05$ (red).

the evolution of a and ω determined. Figures 3 and 4 show the parameters a and ω vs. τ for different values of the damping ratio ζ. Figures 5 and 6 show a plot of the transverse displacement of the beam vs. time for different values of ζ, for a beam made of linear elastic (dashed line) and masonry–like (continuous line) material. Energy a decreases following an exponential law, as does the amplitude A of the motion. Instead, the frequency ω, starting at the initial value ω_0, which is the same as in the absence of damping [9], tends to the linear elastic value π^2. Depending on ζ, the values τ_{1el} and τ_{2el} represent the instants at which this linear elastic value is reached and the solution tends to assume the linear elastic form. At these times the beam will fall entirely in the linear elastic field and ξ_0 will towards $\frac{1}{2}$.

3.2. Forced damped oscillations

Equations (19) and (20) are now taken in their complete form. Before showing an application, some specifications are needed. Firstly, the unimodal assumption (26) involves in some limitations on the form of the forcing term. In particular, a unimodal form of the solution is expected if the beam is subjected to a sinusoidal forcing term whose frequency is close to the fundamental linear elastic one [16, 17]. This is precisely the case we will study, by

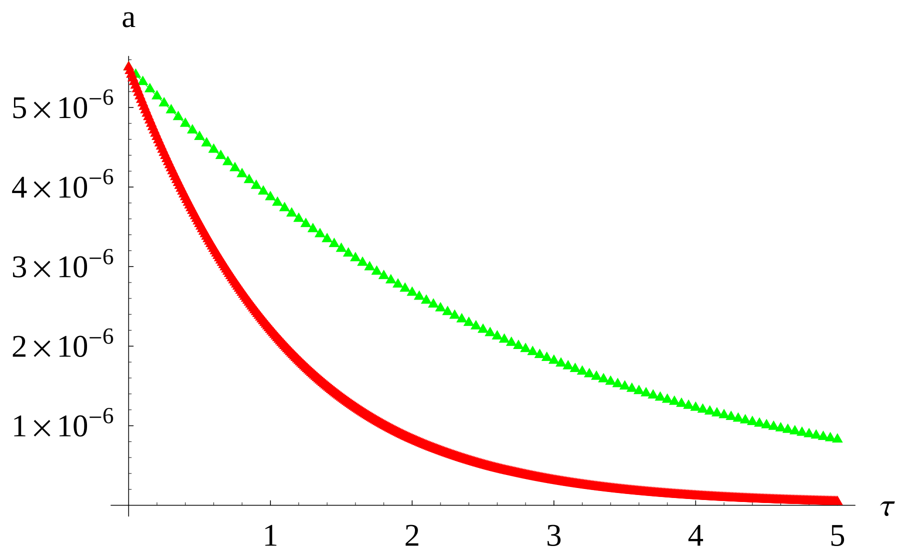


Figure 4: Energy a vs. τ for $\zeta_1 = 0.02$ (green) and $\zeta_2 = 0.05$ (red).

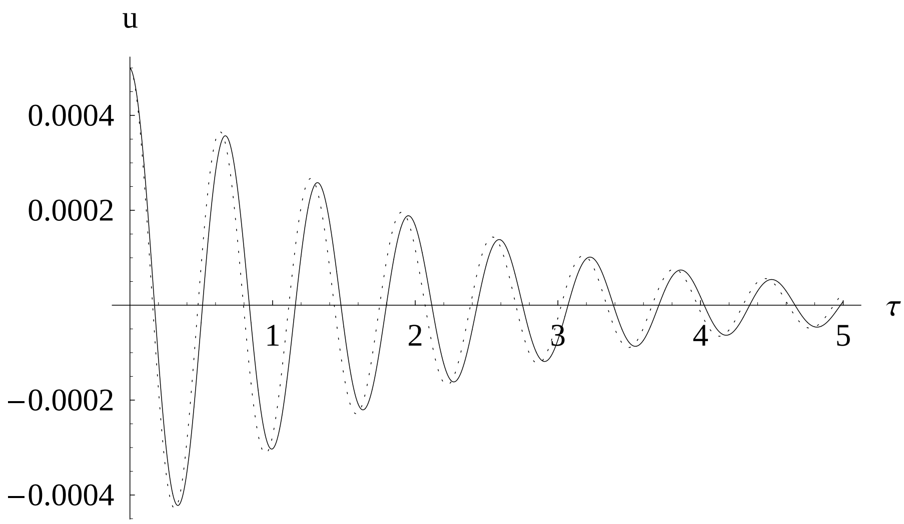


Figure 5: Transverse displacements u of the beam midpoint vs. τ for $\zeta = 0.05$, $\kappa_0 = 0.002$ and $A_0 = 0.0005/\sqrt{2}$ in the linear (dotted) and nonlinear (continuous) case.

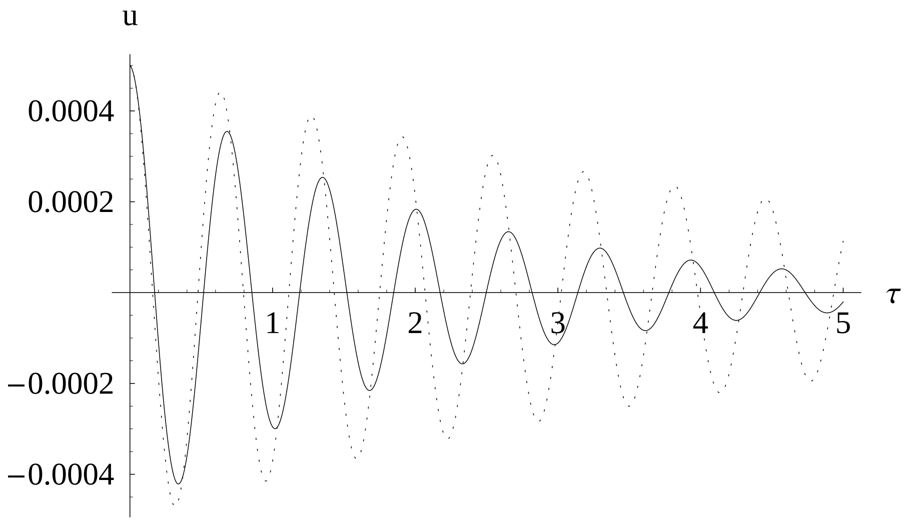


Figure 6: Transverse displacements u of the beam midpoint vs. τ for $\zeta = 0.02$, $\kappa_0 = 0.002$ and $A_0 = 0.0005/\sqrt{2}$ in the linear (dotted) and nonlinear (continuous) case.

considering a forcing term of the form

$$\bar{p} = \bar{k} \sin(\pi^2 \tau + \lambda \tau), \tag{44}$$

where

$$\bar{k} = \int_0^1 k(\xi)\phi(\xi) d\xi \tag{45}$$

and λ is small.

Secondly, the forcing terms in the modulation equations (19) and (20) cannot be dealt with without choosing an "a priori" form for $U(\theta)$. We know that the response over time of nonlinear elastic systems subjected to primary resonance tends to the forcing action frequency and exhibit a phase shift that can be considered a slowly varying function of time [16, 17]. Thus, a possible form for $U(\theta)$ is

$$U(\theta) = A \sin(\theta), \tag{46}$$

where

$$\theta = (\pi^2 + \lambda)\tau - \gamma(T), \tag{47}$$

Equation (44) thus becomes

$$\bar{p} = \bar{k} \sin(\theta + \gamma) \tag{48}$$

and, by introducing (46) and (48) into (19) and (20), the modulation equations take the form

$$\frac{\omega}{2\pi} \oint \frac{dU}{\sqrt{2(a - V(U))}} - 1 = -\frac{\bar{k}}{2} \frac{\partial A}{\partial a} \cos\gamma, \tag{49}$$

$$\frac{a'}{2\pi} \oint \frac{dU}{\sqrt{2(a - V(U))}} = \oint -\frac{\zeta\omega}{\pi} \sqrt{2(a - V(U))}\, dU + \frac{\bar{k}}{2} A \sin\gamma \tag{50}$$

and

$$\omega = \theta' = \pi^2 + \lambda - \gamma'. \tag{51}$$

As for free oscillations, we can find a relation between amplitude A of the motion and energy a of the system

$$a(A) = \frac{1}{2\pi} \int_0^{2\pi} \big(\frac{1}{2}\omega^2 U_\theta^2 + V(U)\big) d\theta, \tag{52}$$

by using expression (46) for U and the approximation $\omega \simeq \pi^2 + \lambda$ for the oscillation frequency. Expression (52) can be inverted to obtain the function $A(a)$ we need to solve the modulation equations (49) and (50).

The stationary values of the parameters can easily be deduced via Equations (49) and (50) by putting $a' = \gamma' = 0$. The system thereby reduces to

$$\frac{\pi^2 + \lambda}{2\pi} \oint \frac{dU}{\sqrt{2(a - V(U))}} - 1 = -\frac{\bar{k}}{2} \frac{\partial A}{\partial a} \cos\gamma, \tag{53}$$

$$\oint \frac{\zeta(\pi^2 + \lambda)}{\pi} \sqrt{2(a - V(U))}\, dU = \frac{\bar{k}}{2} A(a) \sin\gamma \tag{54}$$

and frequency ω tends towards its stationary value

$$\omega = \pi^2 + \lambda. \tag{55}$$

Remark. It seems worthwhile noting that in [7], [8] a different approach has been proposed, by which the assumed form of the displacement is introduced directly in the Lagrangian (5) and integrations performed explicitly. The Euler–Lagrange equations thus lead to the system

$$G(A) + \frac{A\pi}{2}(\lambda - \gamma') + \frac{\bar{k}}{\pi^2}\cos\gamma = 0, \tag{56}$$

$$\frac{A'\pi}{2} + \frac{\zeta\pi^2}{2}A - \frac{\bar{k}}{\pi^2}\sin\gamma = 0, \tag{57}$$

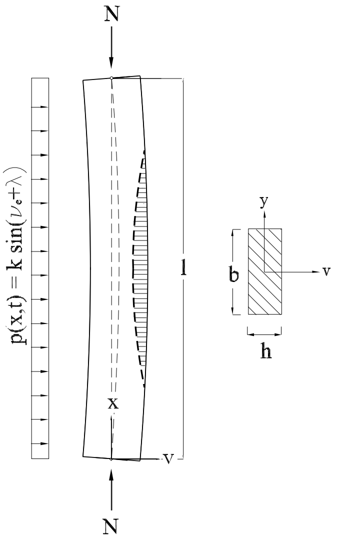
for the slowly varying parameters A and γ, and function $G(A)$ assumes the explicit form

$$\begin{aligned}
G(A) &= \\
&= \frac{A\pi^3}{4} - \frac{A\pi^2}{2}\arcsin\Big(\frac{\kappa_0}{A\pi^2}\Big) + \frac{A\pi^2}{2}\frac{\kappa_0}{\sqrt{A^2\pi^4 - \kappa_0^2}} + \\
&- 2A\pi^4 \int_0^{\frac{1}{\pi^2}\arccos\left(\frac{\kappa_0}{A\pi^2}\right)} \cos^2\left(\pi^2\tau\right)\xi_0\, d\tau - \frac{\kappa_0^3}{2A\pi^2}\frac{1}{\sqrt{A^2\pi^4 - \kappa_0^2}} + \\
&- 10\pi^3\kappa_0 A \int_0^{\frac{1}{\pi^2}\arccos\left(\frac{\kappa_0}{A\pi^2}\right)} \frac{\cos^2(\pi^2\tau)}{\sqrt{A^2\pi^4\cos^2(\pi^2\tau) - \kappa_0^2}}\, d\tau + \\
&+ 8\pi\kappa_0^2 \int_0^{\frac{1}{\pi^2}\arccos\left(\frac{\kappa_0}{A\pi^2}\right)} \sqrt{\frac{\cos(\pi^2\tau)}{\kappa_0 A}}\Big(\int_{\xi_0}^{\frac{1}{2}} \sqrt{\sin\pi\xi}\, d\xi\Big)\, d\tau + \\
&+ \frac{10\kappa_0^3}{A\pi} \int_0^{\frac{1}{\pi^2}\arccos\left(\frac{\kappa_0}{A\pi^2}\right)} \frac{1}{\sqrt{A^2\pi^4\cos^2(\pi^2\tau) - \kappa_0^2}}\, d\tau.
\end{aligned} \tag{58}$$

The results obtained using (56) and (57) and (19) and (20) are in considerably good agreement.

4. Some example applications

Some numerical tests have been performed, using the scheme shown in Figure 7. Three values were chosen for slenderness, with the corresponding section height h equal to 0.30 m, 0.40 m and 0.50 m; two values of damping ratio ζ were considered: 2% and 5%. The beam is subjected to a sinusoidal load of variable amplitude k and frequency $(\nu_e + \lambda)$. For the three slenderness values chosen, the numerical values of the fundamental frequency



Fixed parameters:
$l = 6$ m; $b = 1$ m;
$\rho = 1800$ kg/m^3; $E = 3 \cdot 10^9$ Pa;
$N = 100$ KN.

For $\zeta = 0.02$ and $\lambda = 0$:

h [m]	k [N/m]					
0.30	100	200	300	400	600	800
0.40	100	200	300	400	600	800
0.50	100	200	300	400	600	800

For $\zeta = 0.05$ and $\lambda = 0$:

h [m]	k [N/m]					
0.40	100	200	300	400	600	800

For $\zeta = 0.02$ and $k = 400$ N/m:

h [m]	λ [Hz]							
0.40	-1.5	-1.25	-1	-0.75	-0.5	0	0.25	0.5

Figure 7: Geometry of the beam and data used for the numerical tests.

are $\nu_e(0.3\text{ m}) = 4.9$ Hz, $\nu_e(0.4\text{ m}) = 6.5$ Hz and $\nu_e(0.5\text{ m}) = 8.1$ Hz. For all tests, null initial displacements and velocities have been imposed on the beam. Some tests have been performed for a beam made of a linear elastic material too.

The analytical results have been compared with those obtained via the NOSA–ITACA, in which the masonry–like constitutive equation has been implemented, for both static and dynamic problems. With the aim of optimizing the comparisons between the analytical and numerical results, different kinds of elements have been tested, while varying the number of elements as well. Lastly, the eight–node isoparametric thin–shell element described in [12] was chosen and the beam divided into 120 finite–elements. Some other comparisons can be found in [7] and [9], where the analytical results are tested via the MADY code.

Figures 8 and 9 show the displacements of the beam mid–point $v_{L/2}$ for $h = 0.40$ m, $k = 400$ N/m and different values of the damping ratio ζ. In

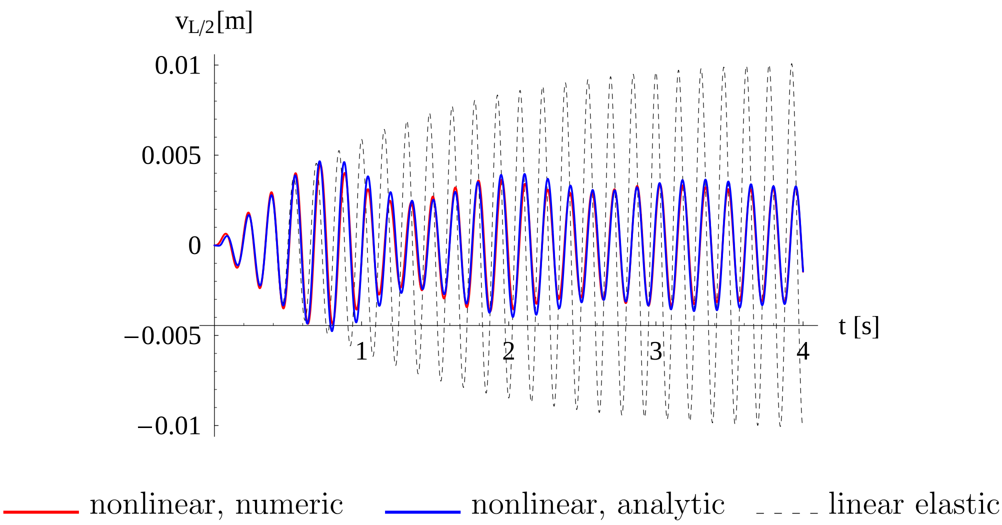


Figure 8: Displacements of the beam mid–section vs. time t for $\zeta = 0.02$, $k = 400$ N/m, $h = 0.4$ m.

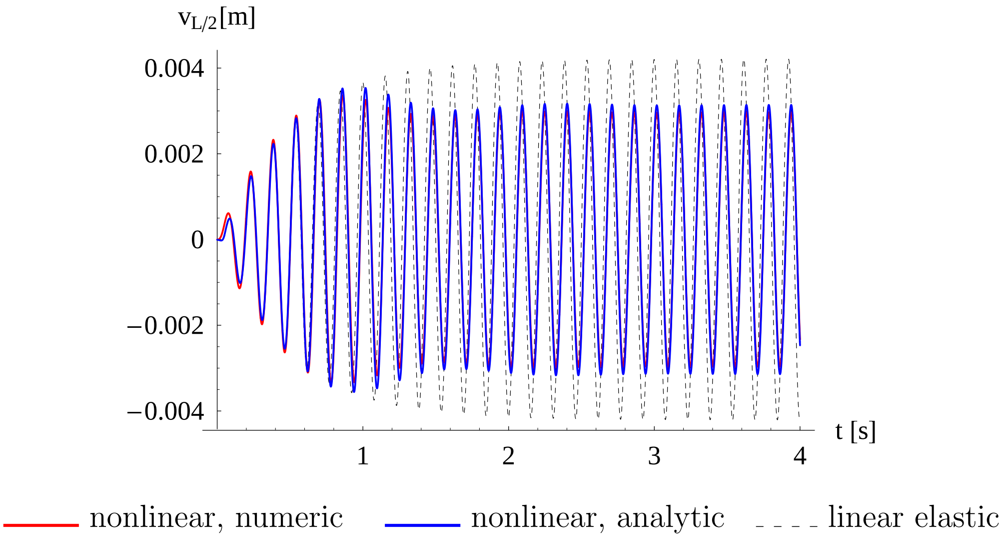


Figure 9: Displacements of the beam mid–section vs. time t for $\zeta = 0.05$, $k = 400$ N/m, $h = 0.4$ m.

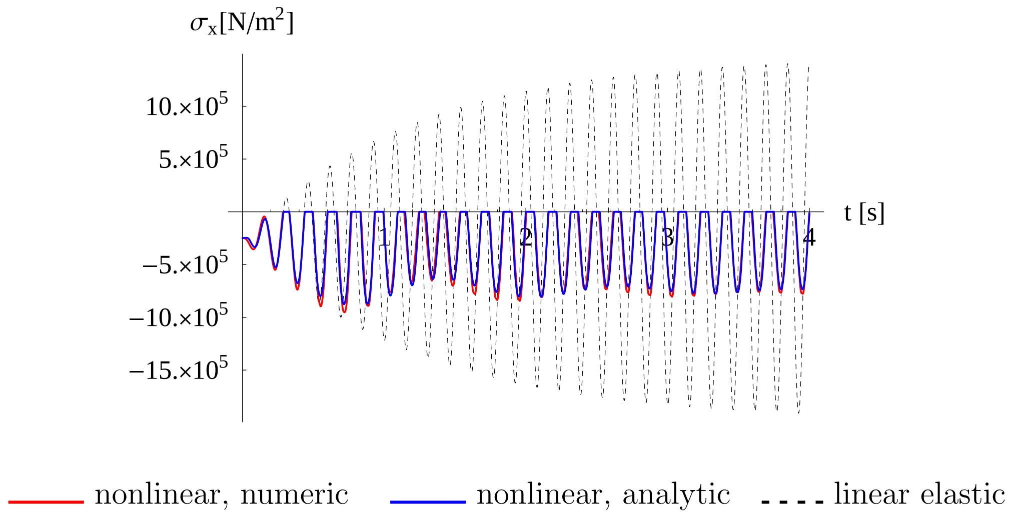


Figure 10: Stress σ_x at the extrados of the beam mid–section vs. time t for $\zeta = 0.02$, $k = 400$ N/m, $h = 0.4$ m.

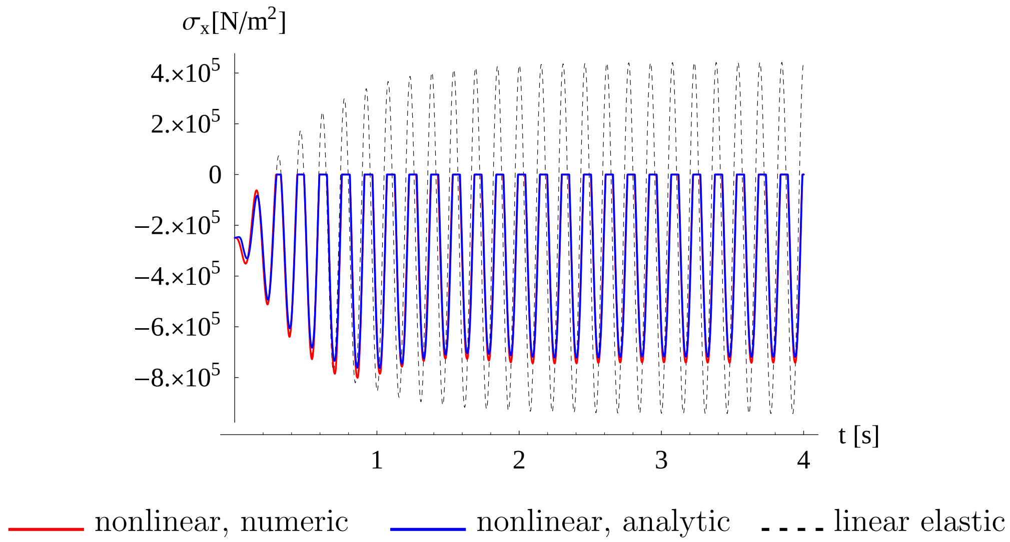


Figure 11: Stress σ_x at the extrados of the beam mid–section vs. time t for $\zeta = 0.05$, $k = 400$ N/m, $h = 0.4$ m.

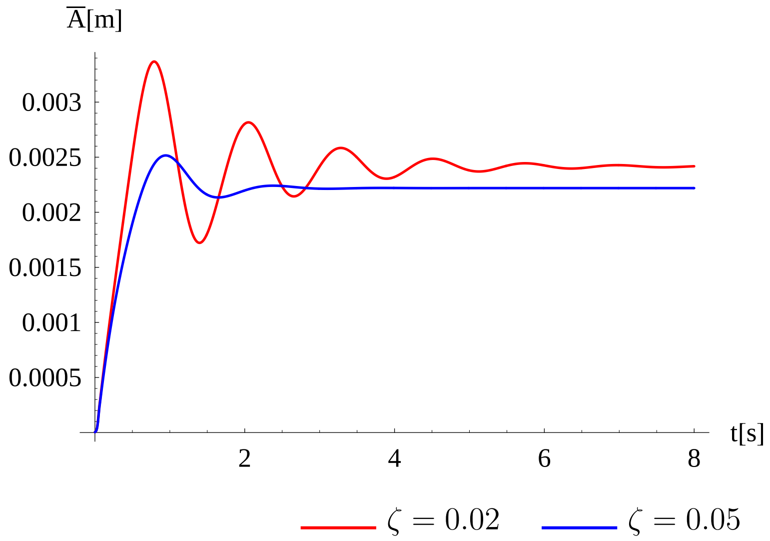


Figure 12: Amplitude $\bar{A}$ of displacements vs. time t for $k = 400$ N/m, $h = 0.4$ m, $\lambda = 0$.

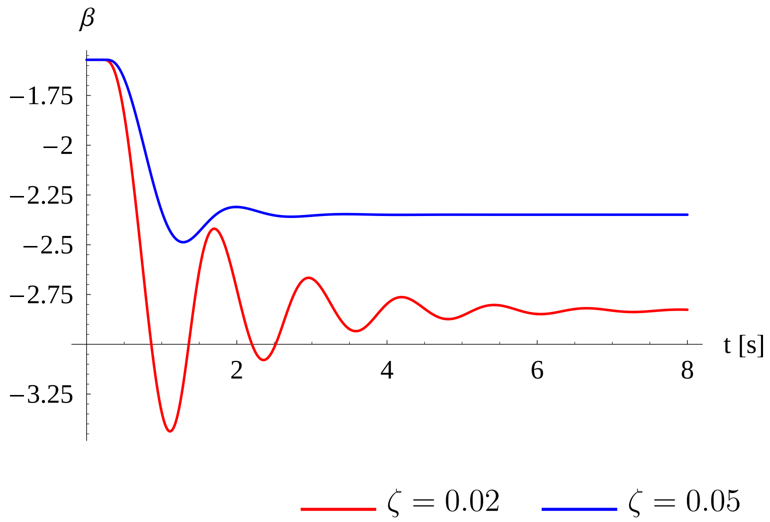


Figure 13: Phase-shift β of displacements vs. time t for $k = 400$ N/m, $h = 0.4$ m, $\lambda = 0$.

both cases, the analytical and numerical results are quite consistent. Note that, after a brief transient stage, the oscillations tend toward stationary behaviour. Figures 10 and 11 instead show the stress σ_x at the extrados of the mid–section vs. t for $h = 0.40$ m, $k = 400$ N/m and different values of damping ratio ζ. Figures 12 and 13 show the behaviour of amplitude A and the phase displacement

$$\beta = \lambda t - \gamma \tag{59}$$

vs. t for $\lambda = 0$, $k = 400$ N/m, $h = 0.4$ m and different values of ζ. When compared with Figures 8 and 9, these figures confirm the slow variation of the parameters A and γ. Note that the curves in Figure 13 start at the same value $\beta = -\frac{\pi}{2}$, corresponding to the linear elastic solution to the problem. Figures 14 and 15 show the stationary amplitude $\bar{A} = \sqrt{2}Al$ vs. k for different values of h and ζ. The nonlinear values are quite different from the corresponding linear ones and all curves tend to exhibit very marked softening behaviour. It is worth noting that the curves related to the different damping values in Figure 15 tend to coincide for large values of k. All the numerical amplitude values are lightly lower than the corresponding analytical ones, that is to say: the numerical tests tend to reveal more softening behaviour. Figure 16 shows the phase displacements β vs. k; the numerical values were obtained using the Fast Fourier Transform. The analytical solution shows two branches. However, for all the numerical solutions, the phase displacements lie on the lower branch, while the horizontal line for $\beta = -\frac{\pi}{2}$ represents the linear elastic solution. Lastly, Figure 17 shows a comparison between the linear elastic frequency response function (dashed curve), the corresponding analytical nonlinear function (continuous curve) and the results of the numerical tests with variable frequency excitations (red curve). The differences between the linear and nonlinear responses are considerable, particularly in the range centred on the linear fundamental frequency. The nonlinear analytical curve presents the typical shift towards low frequencies characteristic of softening systems. Moreover, for excitations of given frequency and amplitude, the curve presents more than one solution, depending on the initial conditions. In our tests, with the chosen initial conditions, the numerical solution presents a jump at about 5.6 Hz, from the upper to the lower branch of the analytical curve.

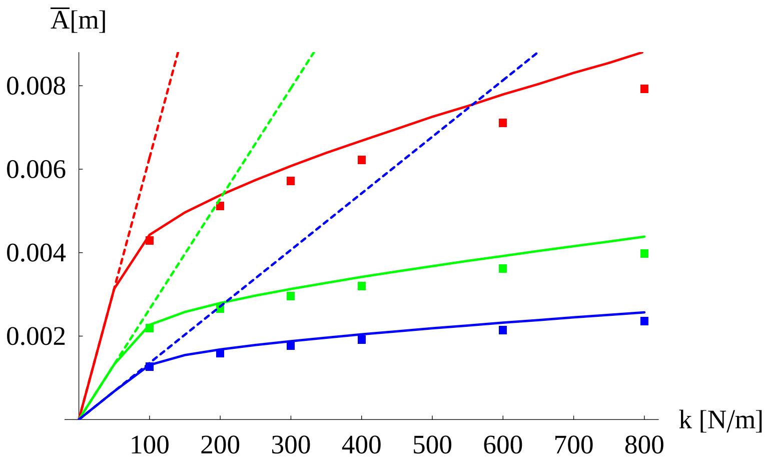


Figure 14: Maximum stationary displacement of the beam vs. k for $\lambda = 0$, $\zeta = 0.02$ and different values of h. ■ nonlinear, numeric ―― nonlinear, analytic - - - - linear elastic.

5. Conclusions

An analytical method has been presented to study the periodic oscillation of masonry beam–columns, under some hypotheses on the geometry and form of the solution. The nonlinear behaviour of masonry has been taken into account by means of a masonry–like constitutive equation expressed in terms of generalized stresses and strains. Some example applications have been shown and the analytical results compared to those obtained via the finite element code NOSA–ITACA. The analytical and numerical results have proven to be consistently in good agreement. The numerical methods enable solving problems for very general conditions of geometry and loading. However, the analytical solutions, albeit limited to some particular cases, provide coincise descriptions of the nonlinear phenomena involved and contribute to a better understanding of the overall behaviour of masonry structures.

A. Appendix

The averaged Lagrangian method has been described in section 2 and applied to masonry–like materials in sections 3 and 4. However, finding a

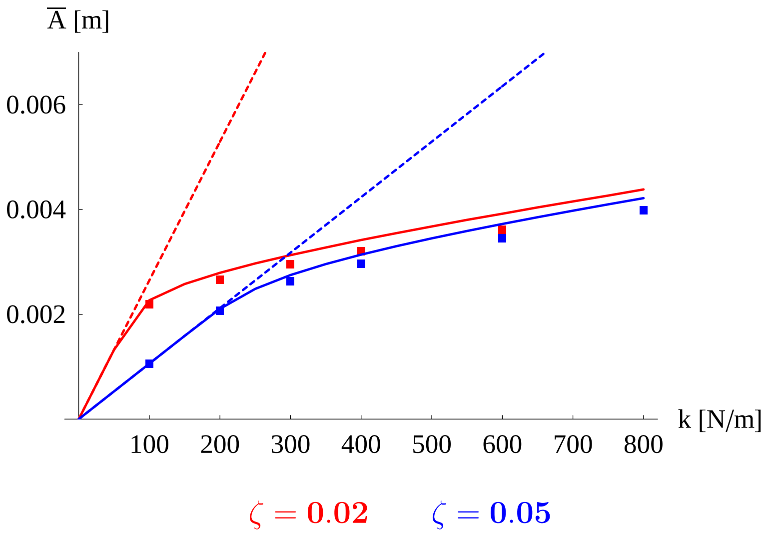


Figure 15: Maximum stationary displacement of the beam vs. k for $\lambda = 0$, $h = 0.4$ m and different values of ζ. ■ nonlinear, numeric —— nonlinear, analytic - - - - linear elastic.

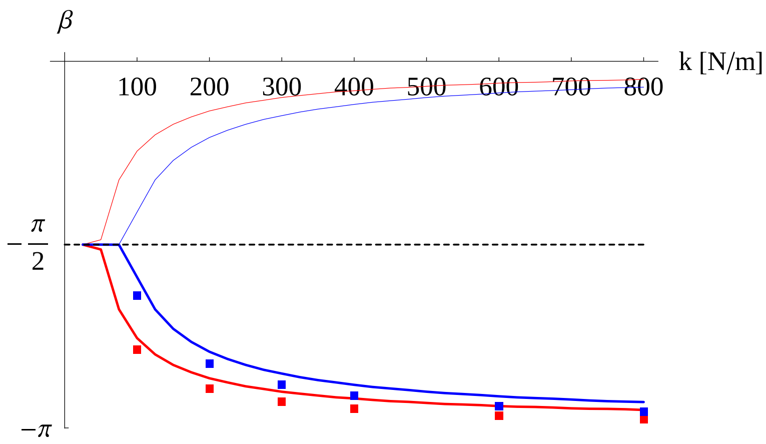
formal justification for this powerful and intuitive method is not an easy matter. In [19] such a justification is furnished via the multiple scales method: Whitham applies the multiple scales or "two–timing" method to the study of nonlinear dispersive waves and demonstrates analogous results by writing the variational equations for the averaged Lagrangian. In this Appendix Whitham's scheme is followed to prove the effectiveness of the averaged Lagrangian method for nonlinear elastic beams subjected to small nonconservative actions.

Firstly, let us use the multiple scales method [16, 19] to find an approximate solution to (9). To this end, if ε is a small dimensionless parameter of the same order of magnitude as the amplitude of the beam's transverse displacement, we can introduce the new variable

$$T = \varepsilon\tau, \tag{60}$$

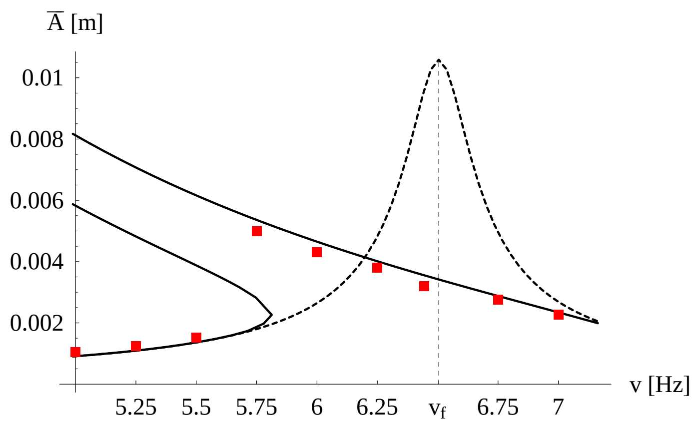
which in same sense measures the "slow" time scale of the problem. We can explicitly split function η defined by (7) into its fast and slow oscillating parts and write

$$\eta = U(\theta, T; \varepsilon), \tag{61}$$



h = 0.3 m **h** = 0.5 m

Figure 16: Stationary values of β vs. k for $\lambda = 0$, $\zeta = 0.02$ and different values of h.
■ nonlinear, numeric ——— nonlinear, analytic - - - - linear elastic.



■ nonlinear, numeric ——— nonlinear, analytic - - - - linear elastic

Figure 17: Maximum stationary displacement of the beam vs. the excitation frequency ν for $\zeta = 0.02$ and $k = 400$ N/m, $h = 0.4$ m.

with

$$\theta = \varepsilon^{-1}\Theta(T), \qquad\qquad \theta_\tau = \omega. \tag{62}$$

In (62) θ represents the phase of the periodic function η, while its derivative θ_τ with respect to time is the oscillation frequency ω. Note that ω is a slowly varying function:

$$\omega = \varepsilon^{-1}\Theta_\tau = \Theta_T. \tag{63}$$

The time derivatives can now be scaled so that

$$\frac{\partial}{\partial\tau} = \omega\frac{\partial}{\partial\theta} + \varepsilon\frac{\partial}{\partial T}. \tag{64}$$

Now, by putting

$$\bar{L}_1 = \frac{\partial\bar{L}}{\partial\eta_\tau}, \quad \bar{L}_2 = \frac{\partial\bar{L}}{\partial\eta}, \tag{65}$$

equation (9) becomes

$$\omega\frac{\partial\bar{L}_1}{\partial\theta} + \varepsilon\frac{\partial\bar{L}_1}{\partial T} - \bar{L}_2 = -\varepsilon\left(2\bar{\mu}\omega\frac{\partial U}{\partial\theta} - \bar{\bar{p}}\right), \tag{66}$$

with $\bar{L}$ given by (10), $\bar{\mu}\,\varepsilon = \mu$, $\bar{\bar{p}}\,\varepsilon = \bar{p}$ and we have neglected the terms of the ε^2 order. In (66) the smallness of the nonconservative terms is expressed explicitly by means of ε.

We can write (66) in the form

$$\frac{\partial}{\partial\theta}(\omega\bar{L}_1\frac{\partial U}{\partial\theta} - \bar{L}) + \varepsilon\frac{\partial}{\partial T}(\bar{L}_1\frac{\partial U}{\partial\theta}) = -\varepsilon\left(2\bar{\mu}\omega\frac{\partial U}{\partial\theta} - \bar{\bar{p}}\right)\frac{\partial U}{\partial\theta}. \tag{67}$$

Indeed, with the help of (64), we have

$$\frac{\partial\bar{L}}{\partial\theta} = \bar{L}_1\frac{\partial}{\partial\theta}(\omega\frac{\partial U}{\partial\theta} + \varepsilon\frac{\partial U}{\partial T}) + \bar{L}_2\frac{\partial U}{\partial\theta} = \bar{L}_1(\omega\frac{\partial^2 U}{\partial\theta^2} + \varepsilon\frac{\partial^2 U}{\partial\theta\partial T}) + \bar{L}_2\frac{\partial U}{\partial\theta}. \tag{68}$$

By introducing (68) into (67) and performing all the calculations we obtain

$$\frac{\partial U}{\partial\theta}\left(\omega\frac{\partial\bar{L}_1}{\partial\theta} + \varepsilon\frac{\partial\bar{L}_1}{\partial T} - \bar{L}_2\right) = -\varepsilon\left(2\bar{\mu}\omega\frac{\partial U}{\partial\theta} - \bar{\bar{p}}\right)\frac{\partial U}{\partial\theta} \tag{69}$$

which is equivalent to (66).

By taking the expressions

$$U = \sum_{n=0}^{\infty} \varepsilon^n \, U^{(n)}, \tag{70}$$

$$\bar{L} = \sum_{n=0}^{\infty} \varepsilon^n \, \bar{L}(U^{(n)}, \omega U_\theta^{(n)}) = \sum_{n=0}^{\infty} \varepsilon^n \, \bar{L}^{(n)}, \tag{71}$$

equation (67) can be expanded into the set

$$\frac{\partial}{\partial\theta}(\omega \bar{L}_1^{(0)} \frac{\partial U^{(0)}}{\partial\theta} - \bar{L}^{(0)}) = 0, \tag{72}$$

$$\frac{\partial}{\partial\theta}(\omega \bar{L}_1^{(1)} \frac{\partial U^{(1)}}{\partial\theta} - \bar{L}^{(1)}) = -\frac{\partial}{\partial T}(\bar{L}_1^{(0)} \frac{\partial U^{(0)}}{\partial\theta}) - 2\mu\omega \left(\frac{\partial U^{(0)}}{\partial\theta}\right)^2 + \bar{\bar{p}} \, \frac{\partial U^{(0)}}{\partial\theta}, \tag{73}$$

where, again, we have neglected terms of the ε^2 order.

Equation (72) yields

$$\omega \bar{L}_1^{(0)} \frac{\partial U^{(0)}}{\partial\theta} - \bar{L}^{(0)} = a(T), \tag{74}$$

where $a(T)$ is a slowly varying parameter related to the energy of the system. By virtue of (14), equation (74) can be expressed as

$$\frac{1}{2}\omega^2 \left(\frac{\partial U^{(0)}}{\partial\theta}\right)^2 + V(U^{(0)}) = a(T). \tag{75}$$

Remembering that U, U_θ are periodic functions with period 2π, the secular condition on U can be imposed on (73) to arrive at

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial T}(\bar{L}_1^{(0)} \frac{\partial U^{(0)}}{\partial\theta}) \, d\theta = \mathscr{D} + \mathscr{P}, \tag{76}$$

with

$$\mathscr{D} = \frac{1}{2\pi}\int_0^{2\pi} -2\mu\omega \left(\frac{\partial U^{(0)}}{\partial\theta}\right)^2 d\theta = \oint -\frac{\zeta\omega}{\pi}\sqrt{2(a-V)}\, dU^{(0)} \tag{77}$$

and

$$\mathscr{P} = \frac{1}{2\pi} \int_0^{2\pi} \bar{\bar{p}}\, dU^{(0)}. \tag{78}$$

Equations (72) and (76) furnish the approximate solution we are looking for: equation (72) provides the form of the solution, while (76) gives us the modulation.

Let us now consider the variational principle

$$\delta \int \frac{1}{2\pi} \int_0^{2\pi} \bar{L}(U, \omega \frac{\partial U}{\partial \theta} + \varepsilon \frac{\partial U}{\partial T})\, d\theta\, dT = 0 \tag{79}$$

where the assumptions (61) and (62) have been introduced in the Lagrangian (13). The explicit separation of θ from T provided by (61) allows us to consider a and ω constant with respect to θ, when performing integration.

In absence of damping and forcing terms variations δU give [6], [19]

$$\omega \frac{\partial}{\partial \theta} \bar{L}_1 + \varepsilon \frac{\partial}{\partial T} \bar{L}_1 - \bar{L}_2 = 0, \tag{80}$$

while variations $\delta\Theta$ (involved through $\omega = \Theta_T$) furnish

$$\frac{\partial^2 \mathscr{L}}{\partial T \partial \omega} = 0 \tag{81}$$

with $\mathscr{L}$ the averaged Lagrangian given by (16).

We can generalize these results by the extended Hamilton principle [15], [16]. Equation (80) becomes then

$$\omega \frac{\partial}{\partial \theta} \bar{L}_1 + \varepsilon \frac{\partial}{\partial T} \bar{L}_1 - \bar{L}_2 = -\varepsilon \left(2\bar{\mu}\omega \frac{\partial U}{\partial \theta} - \bar{\bar{p}} \right) \tag{82}$$

whose right member represents the component along U of the vector of the generalized forces (damping and external forces) acting upon the system. Equation (80) is equivalent to equation (66), which we have obtained by direct manipulation of the motion equation. Instead, equation (81) becomes

$$\frac{\partial^2 \mathscr{L}}{\partial T \partial \omega} = -\frac{1}{2\pi} \int_0^{2\pi} 2\bar{\mu}\omega \left(\frac{\partial U}{\partial \theta} \right)^2 d\theta + \frac{1}{2\pi} \int_0^{2\pi} \bar{\bar{p}} \frac{\partial U}{\partial \theta} d\theta \tag{83}$$

where this time the right member is the component along $\Theta = \varepsilon\, \theta$ of the vector of the averaged generalized forces. By introducing expressions (70) and (71) into (80) and (81) and neglecting terms of order higher than ϵ^2 we obtain

$$\frac{\partial}{\partial\theta}(\omega\bar{L}_1^{(0)}\frac{\partial U}{\partial\theta}^{(0)} - \bar{L}^{(0)}) = 0, \tag{84}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial T}(\bar{L}_1^{(0)}\frac{\partial U}{\partial\theta}^{(0)})\,d\theta = \mathscr{D} + \mathscr{P}. \tag{85}$$

Equations (84) and (85) are indeed the same as (72) and (76).

Acknowledgements

The financial support of the Region of Tuscany, PAR–FAS 2007–2013, to the Project "Tools for the modelling and assessment of the structural behaviour of ancient constructions: the NOSA–ITACA code", is gratefully acknowledged.

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