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Vincenzo Ciancia, CNR-ISTI, Pisa, Italy

Diego Latella, CNR-ISTI, Pisa, Italy

Mieke Massink, CNR-ISTI, Pisa, Italy

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Bisimulation relations, Spatial logics, Logical equivalence

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Istituto di Scienza e Tecnologie dell'Informazione "A. Faedo"
Area della Ricerca CNR di Pisa
Via G. Moruzzi 1
56124 Pisa Italy
http://www.isti.cnr.it

SLCS on Face-poset Models and Bisimilarities on Quasi-discrete Closure Models*

Vincenzo Ciancia[1][0000−0003−1314−0574], Diego Latella[1][0000−0002−3257−9059], and Mieke Massink[1][0000−0001−5089−002X]

Consiglio Nazionale delle Ricerche - ISTI, Pisa, Italy,
{Vincenzo.Ciancia, Diego.Latella, Mieke.Massink}@cnr.it

Abstract. We define $\mathtt{SLCS}_\eta$, a weaker logic than $\mathtt{SLCS}_\gamma$, and we interpret it on face-poset models. We show the relationship between the equivalence induced by the two logics, namely $\equiv_{\mathtt{SLCS}_\gamma}$ and $\equiv_{\mathtt{SLCS}_\eta}$ and bisimilarities of finite closure models proposed in the literature.

Keywords: Bisimulation relations · Spatial logics · Logical equivalence ·

1 Introduction

The topological approach to spatial logics has its origin in the early ideas by McKinsey and Tarski [17], who gave a topological interpretation of the "necessarily" operator of the **S4** modal logic. The approach was extended to consider *Closure Spaces* (CS) [19], a generalisation of topological spaces, covering also discrete spaces such as general graphs, following work by Galton [13, 15] and Smyth and Webster [18], among others. Recent work by Ciancia et al. (see [10, 11]) builds on these theoretical developments using CSs, or better, *Closure Models* (CMs), as the underlying framework for the *Spatial Logic for Closure Spaces* (SLCS). A closure model is composed of a CS together with a valuation function mapping every atomic proposition letter p of a given set into the set of points in the space satisfying p. A spatio-temporal model checker, `topochecker` [9], has been developed for the subclass of finite closure spaces. Moreover, the spatial model-checker `VoxLogicA`[1] [4] has been developed, that is optimised for digital 2D and 3D images, interpreted as a special case of finite closure spaces, and has been applied successfully in the area of medical imaging [4, 3, 1, 2]. However, for the 2D and 3D visualisation of continuous spatial objects, both in medical imaging

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[1] Available from the `VoxLogicA` repository at https://github.com/vincenzoml/VoxLogicA.

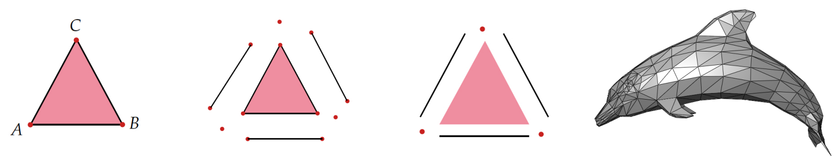
and virtual reality, polyhedral models of *continuous* space are often used. Such spatial models consist of a suitable splitting of the image of an object into areas of different size, known as *meshes*. These include triangular surface meshes or tetrahedral volume meshes (see for example [16]). In [5], an interpretation of SLCS on polyhedral models has been defined. In the sequel, we will refer to it as $\mathtt{SLCS}_\gamma$. Also, a novel notion of bisimilarity for such models, namely *simplicial bisimilarity* has been proposed and the theoretical foundations have been developed for polyhedral model checking, including a global model checking algorithm for $\mathtt{SLCS}_\gamma$ interpreted on *face-poset models*, i.e. discrete and finite representations of polyhedral models. An implementation of the PolyLogicA[1] model-checking tool has been presented. A visualiser for models and model checking results has been developed as well. In [8] $\pm$-*bisimilarity* has been proposed, that is a novel notion of spatial bisimulation for face-poset models. It has also been shown that $\pm$-bisimilarity coincides with the logical equivalence induced by $\mathtt{SLCS}_\gamma$. This result paves the way for the definition of model reduction procedures based on minimisation with respect to $\pm$-bisimilarity, i.e. procedures that are guaranteed to preserve $\mathtt{SLCS}_\gamma$ properties on face-poset models and, finally $\mathtt{SLCS}_\gamma$ properties on the polyhedral models the former represent. Model reduction will contribute to increase efficiency of the model checking algorithms.

In the present report we present $\mathtt{SLCS}_\eta$, a weaker version of $\mathtt{SLCS}_\gamma$. The purpose of investigating weaker logics, and consequently coarser equivalences, is that the latter may provide better minimisation procedures, in the sense of generating smaller models. Face-poset models are a subclass of quasi-discrete closure models (QdCMs).

We also compare the logical equivalences induced by $\mathtt{SLCS}_\gamma$ and $\mathtt{SLCS}_\eta$ with bisimilarities defined on QdCMs that have been investigated in [7, 12]. It turns out that there are bisimilarities on QdCMs — in particular CMC-bisimilarity and CoPa-bisimilarity — that are *stronger* than the equivalence induced by $\mathtt{SLCS}_\gamma$ and so they can be used as a basis for model minimisations that will anyway be correct, although not optimal.

2 Background and Notation

We first introduce some background concepts and related notation. For a function $f : X \to Y$, and subsets $A \subseteq X$ and $B \subseteq Y$, we define $f(A)$ and $f^{-1}(B)$ as $\{f(a) \,|\, a \in A\}$ and $\{a \,|\, f(a) \in B\}$, respectively. The *range* of f is defined as $\mathtt{range}(f) = f(X)$. The *restriction* of f on A is denoted by $f|A$. The set of natural numbers and that of real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. We use the standard interval notation: for $x, y \in \mathbb{R}$ we let $[x, y]$ be the set $\{r \in \mathbb{R} \,|\, x \leq r \leq y\}$, $[x, y) = \{r \in \mathbb{R} \,|\, x \leq r < y\}$ and so on, where $[x, y]$ is equipped with the Euclidean topology inherited from $\mathbb{R}$. We use a similar notation for intervals over $\mathbb{N}$: for $n, m \in \mathbb{N}$ $[m; n]$ denotes the set $\{i \in \mathbb{N} \,|\, m \leq i \leq n\}$, $[m; n)$ denotes the set $\{i \in \mathbb{N} \,|\, m \leq i < n\}$, and similarly for $(m; n]$ and $(m; n)$.



(a) (b) (c) (d)

Fig. 1: (1a) A simplicial complex (actually a simplex itself). (1b) Decomposed into its simplexes as faces. (1c) Partitioned into its cells. (1d) A triangular surface mesh of a dolphin [6].

Definition 1 (Sequences). *Given a set X, a* sequence *over X from x, of* length *$\ell \in \mathbb{N}$, is a total function $s : [0;\ell] \to X$ such that $s(0) = x$. For sequence s of length ℓ, we often use the notation $(x_i)_{i=0}^{\ell}$ where $x_i = s(i)$ for $i \in [0;\ell]$.* •

In the remainder of this section, we recall the main results concerning the interpretation of SLCS on polyhedral models. The interested reader is referred to [5] for a detailed treatment of the subject. Sect. 2.1 below recalls the basic notions of simplex, simplicial complex and polyhedral model. Then, in Sect. 2.2 simplicial bisimilarity and the SLCS interpretation on polyhedral models are briefly reviewed as well as their relationship. The discrete representation of polyhedral models in terms of face-poset models and the SLCS interpretation on the latter is recalled in Sect. 2.3 where their formal relationship is also shown.

2.1 Simplex, Simplicial Complexes and Polyhedra

The notions of simplex, simplicial complex and polyhedron form the basis for geometrical reasoning in a finite setting, amenable to polyhedral model-checking and related techniques. A *simplex* is the convex hull of a set of affinely independent points[2], namely the vertices of the simplex.

Definition 2 (Simplex). *A* simplex *σ of* dimension *d is the convex hull of a finite set $\{\mathbf{v_0}, \ldots, \mathbf{v_d}\} \subseteq \mathbb{R}^m$ of $d+1$ affinely independent points, i.e. $\sigma = \{\lambda_0 \mathbf{v_0} + \ldots + \lambda_d \mathbf{v_d} \,|\, \lambda_0, \ldots, \lambda_d \in [0,1]$ and $\sum_{i=0}^{d} \lambda_i = 1\}$.* •

Note that a simplex is a subset of the ambient space $\mathbb{R}^m$ and so it inherits its topological structure. Given a simplex σ with vertices $\mathbf{v_0}, \ldots, \mathbf{v_d}$, any subset of $\{\mathbf{v_0}, \ldots, \mathbf{v_d}\}$ spans a simplex σ' in turn: we say that σ' is a *face* of σ, written $\sigma' \sqsubseteq \sigma$. Clearly, $\sqsubseteq$ is a partial order relation.

The *relative interior* of a simplex plays a similar role as the notion of "interior" in topology and is defined as follows:

[2] $\mathbf{v_0}, \ldots, \mathbf{v_d}$ are affinely independent if $\mathbf{v_1} - \mathbf{v_0}, \ldots, \mathbf{v_d} - \mathbf{v_0}$ are linearly independent. In particular, this condition implies that $d \leq m$.

Definition 3 (Relative Interior of a Simplex). *Given a simplex σ with vertices $\{\mathbf{v_0}, \ldots, \mathbf{v_d}\}$ the* relative interior $\widetilde{\sigma}$ *of σ is the set* $\{\lambda_0\mathbf{v_0} + \ldots + \lambda_d\mathbf{v_d} \,|\, \lambda_0, \ldots, \lambda_d \in (0,1]$ *and* $\sum_{i=0}^{d} \lambda_i = 1\}$. •

We write $\widetilde{\sigma}' \preceq \widetilde{\sigma}$ whenever $\sigma' \sqsubseteq \sigma$, noting that $\preceq$ is a partial order as well and that $\widetilde{\sigma}' \preceq \widetilde{\sigma}$ if and only if $\widetilde{\sigma}'$ is included in the topological closure of $\widetilde{\sigma}$.

The notion of *simplicial complex* builds upon that of simplex and is the fundamental tool for constructing complex geometrical objects as sets of points in $\mathbb{R}^m$, namely polyhedra, out of simplexes.

Definition 4 (Simplicial Complex and Polyhedron). *A* simplicial complex *K is a finite collection of simplexes of $\mathbb{R}^m$ such that: (i) if $\sigma \in K$ and $\sigma' \sqsubseteq \sigma$ then also $\sigma' \in K$; (ii) if $\sigma, \sigma' \in K$ then $\sigma \cap \sigma' \sqsubseteq \sigma$ and $\sigma \cap \sigma' \sqsubseteq \sigma'$. The* polyhedron *$|K|$ of K is the set-theoretic union of the simplexes in K.* •

Relations $\sqsubseteq$ and $\preceq$ on simplexes are inherited by simplicial complexes: relation $\sqsubseteq$ on simplicial complex K is the union of the face relations on the simplexes composing K, and similarly for $\preceq$. Note that different simplicial complexes can give rise to the same polyhedron and that the set $\widetilde{K} = \{\widetilde{\sigma} \,|\, \sigma \in K \setminus \{\emptyset\}\}$ of non-empty relative interiors of the simplexes of a simplicial complex K forms a partition of polyhedron $|K|$. The elements of $\widetilde{K}$ are called *cells* and $(\widetilde{K}, \preceq)$ is the face-poset of K. By definition of partition, each $x \in |K|$ belongs to a unique cell in the face-poset. We recall that the polyhedron $|K|$ is a subset of the ambient space $\mathbb{R}^m$ and so inherits its topological structure.

Example Fig. 1 shows a triangle as an example of a simplicial complex, and its simplexes in the face relation. The triangle can be partitioned into 7 cells (see Fig. 1c): its interior ($\widetilde{ABC}$, an open triangle), three open segments ($\widetilde{AB}, \widetilde{BC}, \widetilde{AC}$, the sides without endpoints) and the (singletons of the) three vertices ($\widetilde{A}, \widetilde{B}, \widetilde{C}$). Each vertex is a face of two open segments (and of the open triangle itself), and each open segment is a face of the open triangle. The figure shows also a small example of a triangular surface mesh of a dolphin (Fig. 1d). ◇

Paths play a fundamental role in the definition of `SLCS` and are defined below:

Definition 5 (Topological Path). *A topological* path *in a topological space P is a total, continuous function $\pi : [0,1] \to P$.* •

In the polyhedral semantics of `SLCS` proposed in [5], all the points of a polyhedral model that belong to the same cell are required to satisfy the same set of atomic proposition letters. This is reflected in the definition below:

Definition 6 (Polyhedral Model). *For simplicial complex K and set of proposition letters* `AP`*, a* polyhedral model *is a pair $(|K|, V)$ where $V : \mathtt{AP} \to \mathcal{P}(|K|)$ is a valuation function such that, for all $p \in \mathtt{AP}$, $V(p)$ is a union of cells in $\widetilde{K}$.* •

In Figure 2 an example polyhedral model is shown as well as two topological paths. Different proposition letters are shown as different colours in the picture.

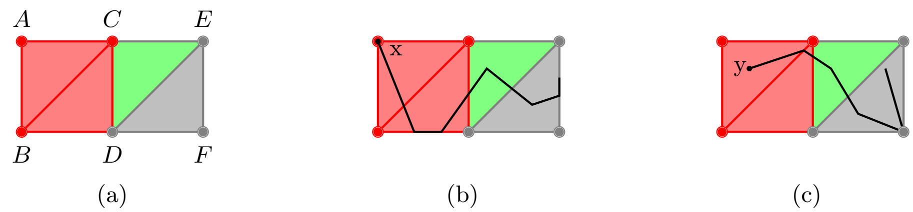


Fig. 2: An example of a polyhedral model (2a) and two paths, one starting from point x (2b) and the other one starting from y (2c). Adapted from [5].

2.2 SLCS on Polyhedral Models

The following definition introduces the variant of SLCS for polyhedral models proposed in [5]. In the present paper, we denote it by $\texttt{SLCS}_\gamma$.

Definition 7 (SLCS on polyhedral models - $\texttt{SLCS}_\gamma$). *The abstract language of $\texttt{SLCS}_\gamma$ is the following:* $\Phi ::= p \mid \neg\Phi \mid \Phi_1 \wedge \Phi_2 \mid \gamma(\Phi_1, \Phi_2)$.
The satisfaction relation of $\texttt{SLCS}_\gamma$ with respect to a given polyhedral model $\mathcal{X} = (|K|, V)$, $\texttt{SLCS}_\gamma$ formula Φ, and $x \in |K|$ is defined recursively on the structure of Φ as follows:

$$
\begin{array}{ll}
\mathcal{X}, x \models p & \Leftrightarrow x \in V(p);\\
\mathcal{X}, x \models \neg\Phi & \Leftrightarrow \mathcal{X}, x \models \Phi \textit{ does not hold};\\
\mathcal{X}, x \models \Phi_1 \wedge \Phi_2 & \Leftrightarrow \mathcal{X}, x \models \Phi_1 \textit{ and } \mathcal{X}, x \models \Phi_2;\\
\mathcal{X}, x \models \gamma(\Phi_1, \Phi_2) & \Leftrightarrow \textit{a topological path } \pi : [0,1] \to |K| \textit{ exists such that } \pi(0) = x,\\
& \quad \mathcal{X}, \pi(1) \models \Phi_2, \textit{ and } \mathcal{X}, \pi(r) \models \Phi_1 \textit{ for all } r \in (0,1).
\end{array}
$$

•

Note that the above definition generalises the classical topological interpretation of the $\Box$ modality as interior and $\Diamond$ as closure. In fact, $\Box\Phi$ is equivalent to $\neg\gamma(\neg\Phi, \texttt{true})$ and, dually, $\Diamond\Phi$ is equivalent to $\gamma(\Phi, \texttt{true})$ (see [5] for details).

Furthermore, note that in order for $\mathcal{X}, x \models \gamma(\Phi_1, \Phi_2)$, it is in general *not* required that x satisfies also Φ_1.

Finally, we point out here that the satisfaction relation does not depend on the specific simplicial complex K, but only on the polyhedron $|K|$ and the valuation of predicate letters V. More precisely, for simplicial complexes K and K' such that $P = |K| = |K'|$ and that give rise to polyhedral models $\mathcal{X} = (|K|, V)$ and $\mathcal{X}' = (|K'|, V)$ the following holds: $\mathcal{X}, x \models \Phi$ if and only if $\mathcal{X}', x \models \Phi$, for every $\texttt{SLCS}_\gamma$ formula Φ and $x \in P$. So, the indication of the specific simplicial complex generating the polyhedral model is not essential, although in the sequel, for notational convenience, we will continue to indicate it explicitly.

Example With reference to model $\mathcal{X}$ of Fig. 2a, it is easy to see that any point in the open segment CD satisfies, for instance, $\gamma(\texttt{green}, \texttt{true})$, and also $\gamma(\texttt{green}, \texttt{red})$ and $\texttt{red} \wedge \gamma(\texttt{green}, \texttt{red})$. ◇

Definition 8 ($\texttt{SLCS}_\gamma$ Logical Equivalence). *Given Polyhedral Model $\mathcal{X} = (|K|, V)$ and $x_1, x_2 \in |K|$ we say that x_1 and x_2 are* logically equivalent *with* *respect to* $\mathtt{SLCS}_\gamma$*, written* $x_1 \equiv^{\mathcal{X}}_{\mathtt{SLCS}_\gamma} x_2$*, if and only if, for all* $\mathtt{SLCS}_\gamma$ *formulas* Φ *the following holds:* $\mathcal{X}, x_1 \models \Phi$ *if and only if* $\mathcal{X}, x_2 \models \Phi$. •

In the sequel, we will refrain from indicating the model $\mathcal{X}$ explicitly in $\equiv^{\mathcal{X}}_{\mathtt{SLCS}_\gamma}$ when it is clear from the context.

2.3 Face-poset Models and SLCS

The following definition characterises the discrete representation of polyhedral models we will use in the rest of the paper (see Fig. 3).

Definition 9 (face-poset model). *Given Polyhedral Model* $\mathcal{X} = (|K|, V)$*, the* face-poset model *of* $\mathcal{X}$ *is the Kripke model* $\mathcal{M}(\mathcal{X}) = (W, \preceq, \mathcal{V})$ *where* $(W, \preceq) = (\widetilde{K}, \preceq)$ *is the face-poset of* K *and* $\widetilde{\sigma} \in \mathcal{V}(p)$ *if and only if* $\widetilde{\sigma} \subseteq V(p)$. •

In the rest of this paper, whenever we say that a Kripke model $\mathcal{F}$ is a face-poset model, we mean that a polyhedral model $\mathcal{X}$ exists such that $\mathcal{F} = \mathcal{M}(\mathcal{X})$.

We now recall the definition of $\pm$-paths introduced in [5]. They faithfully represent, in the face-poset model, topological paths in the polyhedral one.

Definition 10 ($\pm$-path). *Let* $\mathcal{F} = (W, \preceq, \mathcal{V})$ *be a face-poset model and let* $\preceq^{\pm}$ *be the relation* $\preceq \cup \succeq$*. We say that, for* $\ell \in \mathbb{N}$*, sequence* $\pi : [0;\ell] \to W$ *is a* $\pm$-path *(and we indicate it by* $\pi : [0;\ell] \xrightarrow{\pm} W$*) if* $\ell \geq 2$ *and the following holds:* $\pi(0) \preceq \pi(1) \preceq^{\pm} \pi(2) \preceq^{\pm} \ldots \preceq^{\pm} \pi(\ell-1) \succeq \pi(\ell)$. •

The following definition re-interprets $\mathtt{SLCS}_\gamma$ on face-poset models and is based on $\pm$-paths [5].

Definition 11 ($\mathtt{SLCS}_\gamma$ on finite face-posets). *The satisfaction relation of* $\mathtt{SLCS}_\gamma$ *with respect to a given face-poset model* $\mathcal{F} = (W, \preceq, \mathcal{V})$*,* $\mathtt{SLCS}_\gamma$ *formula* Φ*, and* $w \in W$ *is defined recursively on the structure of* Φ*:*

$$
\begin{array}{lll}
\mathcal{F}, w \models p & \Leftrightarrow & w \in \mathcal{V}(p);\\
\mathcal{F}, w \models \neg\Phi & \Leftrightarrow & \mathcal{F}, w \models \Phi \text{ does not hold};\\
\mathcal{F}, w \models \Phi_1 \wedge \Phi_2 & \Leftrightarrow & \mathcal{F}, w \models \Phi_1 \text{ and } \mathcal{F}, w \models \Phi_2;\\
\mathcal{F}, w \models \gamma(\Phi_1, \Phi_2) & \Leftrightarrow & \text{a } \pm\text{-path } \pi : [0;\ell] \xrightarrow{\pm} W \text{ exists such that } \pi(0) = w,\\
 & & \mathcal{F}, \pi(\ell) \models \Phi_2, \text{ and}\\
 & & \mathcal{F}, \pi(i) \models \Phi_1 \text{ for all } i \in (0;\ell).
\end{array}
$$

•

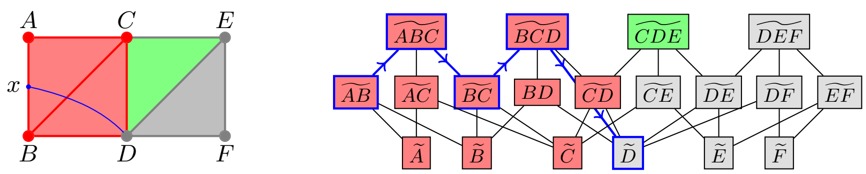
Definition 12 (Logical Equivalence). *Given face-poset model* $\mathcal{F} = (W, \preceq, \mathcal{V})$ *and* $w_1, w_2 \in W$ *we say that* w_1 *and* w_2 *are* logically equivalent *with respect to* $\mathtt{SLCS}_\gamma$*, written* $w_1 \equiv^{\mathcal{F}}_{\mathtt{SLCS}_\gamma} w_2$ *if and only if, for all* $\mathtt{SLCS}_\gamma$ *formulas* Φ *the following holds:* $\mathcal{F}, w_1 \models \Phi$ *if and only if* $\mathcal{F}, w_2 \models \Phi$. •

In the sequel, we will refrain from indicating the model $\mathcal{F}$ explicitly in $\equiv^{\mathcal{F}}_{\mathtt{SLCS}_\gamma}$ when it is clear from the context.

A fundamental result, see [5], follows, where with slight overloading, for $x \in |K|$, we let $\mathcal{M}(x)$ denote the unique cell $\widetilde{\sigma} \in \widetilde{K}$ such that $x \in \widetilde{\sigma}$ (see Fig. 3 for an illustration).

Theorem 1 (Theorem 4.4 of [5]). *Let $\mathcal{X} = (|K|, V)$ a polyhedral model and $\mathcal{M}(\mathcal{X})$ the associated face-poset model as by Definition 9. For all $x \in |K|$ and $\mathtt{SLCS}_\gamma$ formula Φ it holds that $\mathcal{X}, x \models \Phi$ if and only if $\mathcal{M}(\mathcal{X}), \mathcal{M}(x) \models \Phi$.* □

Example With reference to the face-poset model $\mathcal{M}(\mathcal{X})$ of Fig. 3b for polyhedral model $\mathcal{X}$ of Fig. 2a, it is easy to see that cells $\widetilde{C}$ and $\widetilde{CD}$ satisfy $\gamma(\mathtt{green}, \mathtt{true})$, and also $\gamma(\mathtt{green}, \mathtt{red})$ and $\mathtt{red} \wedge \gamma(\mathtt{green}, \mathtt{red})$. ◇



(a) (b)

Fig. 3: (3a) A polyhedral model $\mathcal{X}$ with atomic propositions **red**, **green** and **gray**, and a path from a point x to vertex D. (3b) Hasse diagram of face-poset model $\mathcal{M}(\mathcal{X})$ and a $\pm$-path (in blue) corresponding to the path in $\mathcal{X}$.

3 Weak SLCS on face-poset models

In this section we consider a weaker version of $\mathtt{SLCS}_\gamma$ denoted by $\mathtt{SLCS}_\eta$. The language of the logic is obtained by replacing the reachability operator $\gamma(\Phi_1, \Phi_2)$ with $\eta(\Phi_1, \Phi_2)$. Intuitively, $\eta(\Phi_1, \Phi_2)$ is equivalent to $\Phi_1 \wedge \gamma(\Phi_1, \Phi_2)$.[3]

Definition 13 ($\mathtt{SLCS}_\eta$ on finite face-poset models). *Let $\mathcal{F} = (W, \preceq, \mathcal{V})$ be the face-poset model. Given $w \in W$, satisfaction $\mathcal{F}, w \models \phi$ over $\mathtt{SLCS}_\eta$ formulas ϕ is given by the following inductive clauses:*

$$
\begin{array}{lll}
\mathcal{F}, w \models p & \Leftrightarrow & w \in \mathcal{V}(p); \\
\mathcal{F}, w \models \neg\Phi & \Leftrightarrow & \mathcal{F}, w \not\models \Phi; \\
\mathcal{F}, w \models \Phi_1 \vee \Phi_2 & \Leftrightarrow & \mathcal{F}, w \models \Phi_1 \text{ or } \mathcal{F}, w \models \Phi_2; \\
\mathcal{F}, w \models \eta(\Phi_1, \Phi_2) & \Leftrightarrow & \text{a } \pm\text{-path } \pi : [0; \ell] \xrightarrow{\pm} W \text{ exists such that} \\
& & \pi(0) = w, \\
& & \mathcal{F}, \pi(\ell) \models \Phi_2 \text{ and} \\
& & \mathcal{F}, \pi(i) \models \Phi_1 \text{ for all } i \in [0; \ell).
\end{array}
$$

•

Definition 14 (Logical Equivalence). *Let $\mathcal{F} = (W, \preceq, \mathcal{V})$ be a face-poset model. For all $w_1, w_2 \in W$ we say that w_1 and w_2 are* logically equivalent*,*

[3] Modal operator η relates to γ in a similar way as operator ζ, defined in [7] in the context of quasi-discrete closure spaces, relates to ρ.

written $w_1 \equiv^{\mathcal{F}}_{\mathtt{SLCS}_\eta} w_2$ *if and only if, for all* $\mathtt{SLCS}_\eta$ *formulas* Φ*, the following holds:* $\mathcal{F}, w_1 \models \Phi$ *if and only if* $\mathcal{F}, w_2 \models \Phi$*.* •

In the sequel, we will refrain from indicating the model $\mathcal{F}$ explicitly in $\equiv^{\mathcal{F}}_{\mathtt{SLCS}_\eta}$ when it is clear from the context.

Below, we show that $\mathtt{SLCS}_\eta$ can be encoded into $\mathtt{SLCS}_\gamma$ which implies that the former is weaker than the latter.

Definition 15. *We define the encoding* $\mathcal{E}$ *of* $\mathtt{SLCS}_\eta$ *into* $\mathtt{SLCS}_\gamma$*:*

$$\begin{array}{ll} \mathcal{E}(p) & = p \\ \mathcal{E}(\neg\Phi) & = \neg\mathcal{E}(\Phi) \\ \mathcal{E}(\Phi_1 \wedge \Phi_2) & = \mathcal{E}(\Phi_1) \wedge \mathcal{E}(\Phi_2) \\ \mathcal{E}(\eta(\Phi_1, \Phi_2)) & = \mathcal{E}(\Phi_1) \wedge \gamma(\mathcal{E}(\Phi_1), \mathcal{E}(\Phi_2)) \end{array}$$

•

The following lemma is easily proven by structural induction using the relevant definitions:

Lemma 1. *Let* $\mathcal{F} = (W, \preceq, \mathcal{V})$ *be a face-poset model,* $w \in W$ *and* Φ *a* $\mathtt{SLCS}_\eta$ *formula. Then* $\mathcal{F}, w \models \Phi$ *if and only if* $\mathcal{F}, w \models \mathcal{E}(\Phi)$*.*

Proof. By induction on the structure of Φ. We consider only the case $\eta(\Phi_1, \Phi_2)$. Suppose $\mathcal{F}, w \models \eta(\Phi_1, \Phi_2)$. By definition there is a $\pm$-path π of some length $\ell \geq 2$ such that $\mathcal{F}, \pi(\ell) \models \Phi_2$ and $\mathcal{F}, \pi(i) \models \Phi_1$ for all $i \in [0; \ell)$. By the Induction Hypothesis this is the same to say that $\mathcal{F}, \pi(\ell) \models \mathcal{E}(\Phi_2)$ and $\mathcal{F}, \pi(i) \models \mathcal{E}(\Phi_1)$ for all $i \in [0; \ell)$, i.e. $\mathcal{F}, w \models \mathcal{E}(\Phi_1)$, $\mathcal{F}, \pi(\ell) \models \mathcal{E}(\Phi_2)$ and $\mathcal{F}, \pi(i) \models \mathcal{E}(\Phi_1)$ for all $i \in (0; \ell)$. In other words, we have $\mathcal{F}, w \models \mathcal{E}(\Phi_1) \wedge \gamma(\mathcal{E}(\Phi_1), \mathcal{E}(\Phi_2))$ that, by Definition 15 means $\mathcal{F}, w \models \mathcal{E}(\eta(\Phi_1, \Phi_2))$.

Suppose now $\mathcal{F}, w \models \mathcal{E}(\eta(\Phi_1, \Phi_2))$, i.e. $\mathcal{F}, w \models \mathcal{E}(\Phi_1) \wedge \gamma(\mathcal{E}(\Phi_1), \mathcal{E}(\Phi_2))$, by Definition 15. Since $\mathcal{F}, w \models \gamma(\mathcal{E}(\Phi_1), \mathcal{E}(\Phi_2))$, there is a $\pm$-path π of some length $\ell \geq 2$ such that $\mathcal{F}, \pi(\ell) \models \mathcal{E}(\Phi_2)$ and $\mathcal{F}, \pi(i) \models \mathcal{E}(\Phi_1)$ for all $i \in (0; \ell)$. Using the Induction Hypothesis we know the following holds: $\mathcal{F}, w \models \Phi_1$, $\mathcal{F}, \pi(\ell) \models \Phi_2$, and $\mathcal{F}, \pi(i) \models \Phi_1$ for all $i \in (0; \ell)$, i.e. $\mathcal{F}, \pi(\ell) \models \Phi_2$ and $\mathcal{F}, \pi(i) \models \Phi_1$ for all $i \in [0; \ell)$. So, we get $\mathcal{F}, w \models \eta(\Phi_1, \Phi_2)$. □

A direct consequence of Lemma 1 is that $\mathtt{SLCS}_\eta$ is weaker than $\mathtt{SLCS}_\gamma$.

Theorem 2. *Let* $\mathcal{F} = (W, \preceq, \mathcal{V})$ *be a face-poset model. For all* $w_1, w_2 \in W$ *the following holds: if* $w_1 \equiv_{\mathtt{SLCS}_\gamma} w_2$ *then* $w_1 \equiv_{\mathtt{SLCS}_\eta} w_2$*.* □

It is easy to see that the converse of Theorem 2 does not hold and we leave it to the reader to find a counter-example. Furthermore, it is worth noting that the $\Diamond$ modality, defined as recalled below

$$\mathcal{F}, w \models \Diamond\Phi \Leftrightarrow w' \in W \text{ exists such that } w \preceq w' \text{ and } \mathcal{F}, w' \models \Phi$$

cannot be expressed in $\mathtt{SLCS}_\eta$, while it can be expressed in $\mathtt{SLCS}_\gamma$ since $\Diamond\Phi \equiv \gamma(\Phi, \mathtt{true})$.

4 Face-poset models as quasi-discrete closure models

Face-poset models can be seen as a special case of quasi-discrete closure models. Consequently, bisimilarities defined on (quasi-discrete) closure models can be used as a basis for reducing the size of face-poset models. In [7, 12] CM-bisimilarity, CMC-bisimilarity and CoPa-bisimilarity have proposed for quasi-discrete closure models.

Below, we recall the basic notions concerning (quasi-discrete) closure models. We also recall a definition of CM-bisimilarity, the definition of CMC-bisimilarity and a definition of CoPa-bisimilarity.[4]

Then, in the rest of the section, we show the relationship between the above mentioned bisimilarities and $\equiv_{\mathtt{SLCS}_\gamma}$.

Definition 16 (Closure Space – CS). *A* closure space *is a pair* $(X, \mathcal{C})$ *where* X *is a set (of* points*) and* $\mathcal{C} : \mathcal{P}(X) \to \mathcal{P}(X)$ *is the* closure operator*, i.e. a function satisfying the following axioms: (i)* $\mathcal{C}(\emptyset) = \emptyset$*; (ii)* $A \subseteq \mathcal{C}(A)$ *for all* $A \subseteq X$*; and (iii)* $\mathcal{C}(A_1 \cup A_2) = \mathcal{C}(A_1) \cup \mathcal{C}(A_2)$ *for all* $A_1, A_2 \subseteq X$*.* •

It is worth pointing out that CSs are a generalisation of topological spaces. In fact, the latter coincide with CSs that satisfy the *idempotence* axiom, i.e. $\mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A)$ for all $A \subseteq X$.

Definition 17 (Quasi-discrete CS – QdCS). *A* quasi-discrete closure space *is a CS* $(X, \mathcal{C})$ *such that for each* $A \subseteq X$ *it holds that* $\mathcal{C}(A) = \bigcup_{x \in A} \mathcal{C}(\{x\})$*.* •

Every CS $(X, \mathcal{C})$ such that X is a finite set is a QdCS. Given a relation $R \subseteq X \times X$, define the function $\mathcal{C}_R : \mathcal{P}(X) \to \mathcal{P}(X)$ as follows: for all $A \subseteq X$, $\mathcal{C}_R(A) = A \cup \{x \in X \mid \exists a \in A \text{ s.t. } a\,R\,x\}$. It is easy to see that, for any R, $\mathcal{C}_R$ satisfies all the axioms of Definition 16 and so $(X, \mathcal{C}_R)$ is a CS. The following theorem is a standard result in the theory of CSs [14].

Theorem 3. *A CS* $(X, \mathcal{C})$ *is quasi-discrete if and only if there is a relation* $R \subseteq X \times X$ *such that* $\mathcal{C} = \mathcal{C}_R$*.* □

In the sequel, we consider only finite CSs. We let $\overrightarrow{\mathcal{C}}$ denote $\mathcal{C}_R$ and, similarly, $\overleftarrow{\mathcal{C}}$ denote $\mathcal{C}_{R^{-1}}$.

Definition 18 (Finite path). *A finite path in a finite CS* $(X, \mathcal{C})$ *is a total function* $\pi : [0; \ell] \to X$*, for some* $\ell \in \mathbb{N}$*, such that* $\pi(i+1) \in \mathcal{C}(\{\pi(i)\})$ *for all* $i \in [0; \ell)$*.* •

Given a QdCS $(X, \overrightarrow{\mathcal{C}})$ and a path $\pi : [0; \ell] \to X$, we call ℓ the *length* of π and often use the sequence notation $(x_i)_{i=0}^{\ell}$, where $x_i = \pi(i)$ for all $i \in [0; \ell]$ (see Definition 1). More precisely, we say that $(x_i)_{i=0}^{\ell}$ is a *forward path from* x_0

[4] More specifically, the definition of CoPa-bisimilarity we report here is that proposed in [7]. In [12] an alternative definition has been proposed that is more intuitive and has been shown to be equivalent to the original one, used in [7].

if $x_{i+1} \in \overrightarrow{\mathcal{C}}(x_i)$ for $i \in [0;\ell)$ and, similarly, we say that it is a *backward path from* x_0 if $x_{i+1} \in \overleftarrow{\mathcal{C}}(x_i)$ for $i \in [0;\ell)$.

Given a set AP of *atomic proposition letters* the notion of closure *model* (CM for short) is the expected one:

Definition 19 (Closure model – CM). *A* closure model *is a tuple* $\mathcal{G} = (X, \mathcal{C}, \mathcal{V})$, *with* $(X, \mathcal{C})$ *a CS, and* $\mathcal{V} : \mathtt{AP} \to \mathcal{P}(X)$ *the* valuation function, *assigning to each* $p \in \mathtt{AP}$ *the set of points where* p *holds.* •

All definitions for CSs also apply to CMs; thus, a *quasi-discrete closure model* (QdCM for short) is a CM $\mathcal{G} = (X, \mathcal{C}, \mathcal{V})$ where $(X, \mathcal{C})$ is a QdCS. For a closure model $\mathcal{G} = (X, \mathcal{C}, \mathcal{V})$ we often write $x \in \mathcal{G}$ when $x \in X$. Similarly, we speak of paths in $\mathcal{G}$ meaning paths in $(X, \mathcal{C})$.

Clearly, any face-poset model characterises the associated finite CM in the obvious way, as follows: the CM associated to $(W, \preceq, \mathcal{V})$ is $(W, \mathcal{C}_{\preceq}, \mathcal{V})$.

Definition 20 (CM-bisimilarity - $\rightleftharpoons_{\mathtt{CM}}$). *Given a QdCM* $\mathcal{G} = (X, \overrightarrow{\mathcal{C}}, \mathcal{V})$, *a symmetric relation* $B \subseteq X \times X$ *is a CM-bisimulation for* $\mathcal{G}$ *if, whenever* $(x_1, x_2) \in B$, *the following holds:*

1. *for all* $p \in \mathtt{AP}$ *we have* $x_1 \in \mathcal{V}(p)$ *in and only if* $x_2 \in \mathcal{V}(p)$;
2. *for all* $x_1' \in \overrightarrow{\mathcal{C}}(x_1)$, *there is* $x_2' \in \overrightarrow{\mathcal{C}}(x_2)$ *such that* $(x_1', x_2') \in B$;

Two points $x_1, x_2 \in X$ *are called* CM-bisimilar *in* $\mathcal{G}$ *if* $x_1 \, B \, x_2$ *for some CM-bisimulation* B *for* $\mathcal{G}$. *Notation,* $x_1 \rightleftharpoons_{\mathtt{CM}} x_2$. •

Definition 21 (CMC-bisimilarity - $\rightleftharpoons_{\mathtt{CMC}}$). *Given a QdCM* $\mathcal{G} = (X, \overrightarrow{\mathcal{C}}, \mathcal{V})$, *a symmetric relation* $B \subseteq X \times X$ *is a CMC-bisimulation for* $\mathcal{G}$ *if, whenever* $(x_1, x_2) \in B$, *the following holds:*

1. *for all* $p \in \mathtt{AP}$ *we have* $x_1 \in \mathcal{V}(p)$ *in and only if* $x_2 \in \mathcal{V}(p)$;
2. *for all* $x_1' \in \overrightarrow{\mathcal{C}}(x_1)$ *there is* $x_2' \in \overrightarrow{\mathcal{C}}(x_2)$ *such that* $(x_1', x_2') \in B$;
3. *for all* $x_1' \in \overleftarrow{\mathcal{C}}(x_1)$ *there is* $x_2' \in \overleftarrow{\mathcal{C}}(x_2)$ *such that* $(x_1', x_2') \in B$.

Two points $x_1, x_2 \in X$ *are called* CMC-bisimilar *in* $\mathcal{G}$ *if* $x_1 \, B \, x_2$ *for some CMC-bisimulation* B *for* $\mathcal{M}$. *Notation,* $x_1 \rightleftharpoons_{\mathtt{CMC}} x_2$. •

CMC-bisimilarity is the largest CMC-bisimulation. In [7, 12] it has also been shown that CMC-bisimilarity is strictly stronger than CM-bisimilarity, as one would expect.

Definition 22 (CoPa-bisimilarity - $\rightleftharpoons_{\mathtt{CoPa}}$). *Given QdCM* $\mathcal{G} = (X, \overrightarrow{\mathcal{C}}, \mathcal{V})$, *a symmetric relation* $B \subseteq X \times X$ *is a* CoPa-bisimulation for $\mathcal{G}$ *if, whenever* $B(x_1, x_2)$, *the following holds:*

1. *for all* $p \in \mathtt{AP}$ *we have* $x_1 \in \mathcal{V}(p)$ *in and only if* $x_2 \in \mathcal{V}(p)$;
2. *for each forward path* $\pi_1 = (x_i')_{i=0}^{\ell_1}$ *from* x_1 *such that* $B(\pi_1(i), x_2)$ *for all* $i \in [0; \ell_1)$ *there is a forward path* $\pi_2 = (x_j'')_{j=0}^{\ell_2}$ *from* x_2 *such that the following holds:* $B(x_1, \pi_2(j))$ *for all* $j \in [0; \ell_2)$ *and* $B(\pi_1(\ell_1), \pi_2(\ell_2))$;

3. *for each backward path* $\pi_1 = (x'_i)_{i=0}^{\ell_1}$ *from* x_1 *such that* $B(\pi_1(i), x_2)$ *for all* $i \in [0;\ell_1)$ *there is a backward path* $\pi_2 = (x''_j)_{j=0}^{\ell_2}$ *from* x_2 *such that the following holds:* $B(x_1, \pi_2(j))$ *for all* $j \in [0;\ell_2)$ *and* $B(\pi_1(\ell_1), \pi_2(\ell_2))$.

Two points $x_1, x_2 \in X$ *are called* CoPa-bisimilar *in* $\mathcal{G}$ *if* $x_1 \, B \, x_2$ *for some CoPa-bisimulation* B *for* $\mathcal{G}$. *Notation,* $x_1 \rightleftharpoons_{\mathtt{CoPa}} x_2$. •

Although, in general, CMC-bisimilarity is stronger than CoPa-bisimilarity, it easy to prove the following

Theorem 4. *Let* $\mathcal{G} = (X, \overrightarrow{\mathcal{C}}, \mathcal{V})$ *a QdCM with* $\overrightarrow{\mathcal{C}} = \mathcal{C}_R$*, for some non-empty binary relation* $R \subseteq X \times X$*. The following holds: if* R *is a partial order, then CoPa-bisimilarity on* $\mathcal{G}$ *coincides with CMC-bisimilarity.*

Proof. We already know that $\rightleftharpoons_{\mathtt{CMC}} \subseteq \rightleftharpoons_{\mathtt{CoPa}}$ (See Proposition 2 of [7]). In the sequel we show that $\rightleftharpoons_{\mathtt{CoPa}} \subseteq \rightleftharpoons_{\mathtt{CMC}}$ and we do this by showing that $\rightleftharpoons_{\mathtt{CoPa}}$ is a CMC-bisimulation.

Suppose $x_1 \rightleftharpoons_{\mathtt{CoPa}} x_2$. It is straightforward to check that the first condition of Definition 21 is satisfied.

Let x'_1 be any element of $\mathcal{C}_R(\{x_1\})$. Consider the forward path (x_1, x'_1) from x_1. Since $x_1 \rightleftharpoons_{\mathtt{CoPa}} x_2$, there is a forward path π from x_2 of some length ℓ such that $\pi(j) \rightleftharpoons_{\mathtt{CoPa}} x_1$ for all $j \in [0;\ell)$ and $\pi(\ell) \rightleftharpoons_{\mathtt{CoPa}} x'_1$. Furthermore, since R is a partial order, we also have $x_2 \, R \, \pi(\ell)$. But then, by definition of $\mathcal{C}_R$, we get that there is $x'_2 = \pi(\ell) \in \mathcal{C}_R(\{x_2\})$ such that $x'_1 \rightleftharpoons_{\mathtt{CoPa}} x'_2$. Thus $\rightleftharpoons_{\mathtt{CoPa}}$ satisfies the second condition of Definition 21.

The proof regarding the third condition is similar. □

The following theorem shows that logical equivalence w.r.t. $\mathtt{SLCS}_\gamma$ implies CM-bisimilarity.

Theorem 5. *Let* $\mathcal{F} = (W, \preceq, \mathcal{V})$ *be a face-poset model. For all* $w_1, w_2 \in W$ *the following holds: if* $w_1 \equiv_{\mathtt{SLCS}_\gamma} w_2$ *then* $w_1 \rightleftharpoons_{\mathtt{CM}} w_2$.

Proof. In this proof we use the notation introduced below. Let, for $w_1, w_2 \in W$, the $\mathtt{SLCS}_\gamma$ formula δ_{w_1,w_2} be such that if $w_1 \equiv_{\mathtt{SLCS}_\gamma} w_2$, then δ_{w_1,w_2} is $\mathtt{true}$, otherwise, let Φ_{w_1,w_2} be a formula that distinguishes w_1 from w_2, in particular let $\mathcal{F}, w_1 \models \Phi_{w_1,w_2}$ and $\mathcal{F}, w_2 \not\models \Phi_{w_1,w_2}$ and set δ_{w_1,w_2} to Φ_{w_1,w_2}. Put $\chi(w) = \bigwedge_{w' \in W} \delta_{w,w'}$. It is easy to see that, for $w_1, w_2 \in W$, it holds that

$$\mathcal{F}, w_2 \models \chi(w_1) \text{ if and only if } w_1 \equiv_{\mathtt{SLCS}_\gamma} w_2. \tag{1}$$

In fact, suppose $w_1 \not\equiv_{\mathtt{SLCS}_\gamma} w_2$, then we have $\mathcal{F}, w_2 \not\models \delta_{w_1,w_2}$, and so $\mathcal{F}, w_2 \not\models \bigwedge_{w \in W} \delta_{w_1,w}$. If, instead, $w_1 \equiv_{\mathtt{SLCS}_\gamma} w_2$, then we have: $\delta_{w_1,w_1} \equiv \delta_{w_1,w_2} \equiv \mathtt{true}$ by definition, since $w_1 \equiv_{\mathtt{SLCS}_\gamma} w_1$ and $w_1 \equiv_{\mathtt{SLCS}_\gamma} w_2$. Moreover, for any other w, we have that, in any case, $\mathcal{F}, w_1 \models \delta_{w_1,w}$ holds and since $w_1 \equiv_{\mathtt{SLCS}_\gamma} w_2$, also $\mathcal{F}, w_2 \models \delta_{w_1,w}$ holds. So, in conclusion, $\mathcal{F}, w_2 \models \bigwedge_{w \in W} \delta_{w_1,w}$.

We show that $\equiv_{\mathtt{SLCS}_\gamma}$ is a CM-bisimulation relation. Suppose $w_1 \equiv_{\mathtt{SLCS}_\gamma} w_2$. The first condition of Definition 20 follows directly from $w_1 \equiv_{\mathtt{SLCS}_\gamma} w_2$. Below we show that the second condition of Definition 20 is satisfied. Let $w_1' \in \overrightarrow{\mathcal{C}}(\{w_1\})$. By definition of $\overrightarrow{\mathcal{C}}$ we know $w_1 \preceq w_1'$ and, by definition, of γ and (1) on page 11, we know that $\mathcal{F}, w_1 \models \gamma(\chi(w_1'), \mathtt{true})$. Since $w_1 \equiv_{\mathtt{SLCS}_\gamma} w_2$, we also have that $\mathcal{F}, w_2 \models \gamma(\chi(w_1'), \mathtt{true})$. By definition of γ, this means that w_2' exists such that $w_2 \preceq w_2'$ and $\mathcal{F}, w_2' \models \chi(w_1')$. By definition of $\overrightarrow{\mathcal{C}}$ we get $w_2' \in \overrightarrow{\mathcal{C}}(\{w_2\})$. Furthermore $w_2' \equiv_{\mathtt{SLCS}_\gamma} w_1'$, since $\mathcal{F}, w_2' \models \chi(w_1')$. Thus there is $w_2' \in \overrightarrow{\mathcal{C}}(\{w_2\})$ such that $w_1' \equiv_{\mathtt{SLCS}_\gamma} w_2'$. □

Remark 1. Note that the converse of Theorem 5 does not hold, as shown by the model $\mathcal{F}$ of Figure 4 below. Clearly, we have that $\widetilde{AB} \rightleftharpoons_{\mathtt{CM}} \widetilde{BC}$, but we also have

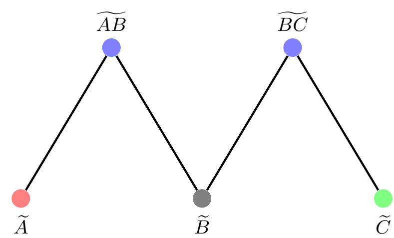


Fig. 4: A face-poset model

$\mathcal{F}, \widetilde{AB} \models \gamma(\mathtt{blue}, \mathtt{red})$ whereas $\mathcal{F}, \widetilde{BC} \not\models \gamma(\mathtt{blue}, \mathtt{red})$. ◇

The following theorem paves the way to performing model checking on models reduced modulo CMC-bisimilarity.

Theorem 6. *Let $\mathcal{F} = (W, \preceq, \mathcal{V})$ be a finite face-poset model. For all $s, t \in W$ the following holds: if $s \rightleftharpoons_{\mathtt{CMC}} t$ then $s \equiv_{\mathtt{SLCS}_\gamma} t$.*

Proof. Suppose $s \rightleftharpoons_{\mathtt{CMC}} t$ and $\mathcal{F}, s \models \Phi$. We proceed by induction on Φ for showing that $\mathcal{F}, t \models \Phi$. By symmetry of $\rightleftharpoons_{\mathtt{CMC}}$ we also get that if $\mathcal{F}, t \models \Phi$ then $\mathcal{F}, s \models \Phi$. We show only the case $\gamma(\Phi_1, \Phi_2)$, the others being straightforward.

Suppose $\mathcal{F}, s \models \gamma(\Phi_1, \Phi_2)$. Then there is $\pi_s : [0; \ell] \xrightarrow{\pm} W$ s.t. $\pi_s(0) = s$, $\mathcal{F}, \pi_s(\ell) \models \Phi_2$ and $\mathcal{F}, \pi_s(i) \models \Phi_1$ for all $i \in (0, \ell)$.

We build $\pi_t : [0; \ell] \xrightarrow{\pm} W$ as follows:

1. we let $\pi_t(0) = t$; recall that $t \rightleftharpoons_{\mathtt{CMC}} s$, and so $\pi_t(0) \rightleftharpoons_{\mathtt{CMC}} \pi_s(0)$;
2. for $j \in [0; \ell)$:
   - If $\pi_s(j) \preceq \pi_s(j+1)$, assuming $\pi_t(j) \rightleftharpoons_{\mathtt{CMC}} \pi_s(j)$, we let $\pi_t(j+1) = v$, where $v \in \overrightarrow{\mathcal{C}}(\{\pi_t(j)\})$ and $v \rightleftharpoons_{\mathtt{CMC}} \pi_s(j+1)$. Note that such a v exists by Lemma 2 below, since $\pi_s(j) \preceq \pi_s(j+1)$ and $\pi_t(j) \rightleftharpoons_{\mathtt{CMC}} \pi_s(j)$. Moreover, $\pi_t(j) \preceq \pi_t(j+1)$ by definition of $\overrightarrow{\mathcal{C}}$ since $\pi_t(j+1) \in \overrightarrow{\mathcal{C}}(\{\pi_t(j)\})$;

- If $\pi_s(j) \succeq \pi_s(j+1)$, assuming $\pi_t(j) \rightleftharpoons_{\mathtt{CMC}} \pi_s(j)$, we let $\pi_t(j+1) = w$ where $w \in \overleftarrow{\mathcal{C}}(\{\pi_t(j)\})$ and $w \rightleftharpoons_{\mathtt{CMC}} \pi_s(j{+}1)$. Note that such a w exists by Lemma 2 below, since $\pi_s(j) \succeq \pi_s(j{+}1)$ and $\pi_t(j) \rightleftharpoons_{\mathtt{CMC}} \pi_s(j)$. Moreover, $\pi_t(j) \succeq \pi_t(j+1)$ by definition of $\overleftarrow{\mathcal{C}}$ since $\pi_t(j+1) \in \overleftarrow{\mathcal{C}}(\{\pi_t(j)\})$.

It is easy to see that the above definition is a good definition of π_t. In particular, we have that, for $i \in [0;\ell]$, $\pi_s(i) \rightleftharpoons_{\mathtt{CMC}} \pi_t(i)$; in fact, we have that:

- $\pi_s(0) \rightleftharpoons_{\mathtt{CMC}} \pi_t(0)$ by hypothesis and,
- at each step i of the procedure, if $\pi_s(i) \rightleftharpoons_{\mathtt{CMC}} \pi_t(i)$, it is guaranteed, by construction, that $\pi_s(i+1) \rightleftharpoons_{\mathtt{CMC}} \pi_t(i+1)$.

Furthermore, since $\pi_s(0) \preceq \pi_s(1)$, $\pi_s(\ell-1) \succeq \pi_s(\ell)$, $\pi_s(i) \preceq^{\pm} \pi_s(i+1)$ for all $i \in (0;\ell{-}1)$ and, by construction, $\pi_t(i) \preceq \pi_t(i{+}1)$ if and only if $\pi_s(i) \preceq \pi_s(i{+}1)$, and $\pi_t(i) \succeq \pi_t(i{+}1)$ if and only if $\pi_s(i) \succeq \pi_s(i{+}1)$, it follows that π_t is a $\pm$-path rooted in t.

Using the I.H. we get $\mathcal{F}, \pi_t(\ell) \models \Phi_2$ and $\mathcal{F}, \pi_t(i) \models \Phi_1$ for all $i \in (0,\ell)$. So, finally, we have $\mathcal{F}, t \models \gamma(\Phi_1, \Phi_2)$.

It is easy to see that the converse of Theorem 6 does not hold.

Lemma 2. *Let $\mathcal{F} = (W, \preceq, \mathcal{V})$ be a face-poset model. For all $s, s', t \in W$ such that $s \rightleftharpoons_{\mathtt{CMC}} t$ the following holds:*

- *if $s \preceq s'$, then there is $t' \in \overrightarrow{\mathcal{C}}(\{t\})$ such that $s' \rightleftharpoons_{\mathtt{CMC}} t'$;*
- *if $s \succeq s'$, then there is $t' \in \overleftarrow{\mathcal{C}}(\{t\})$ such that $s' \rightleftharpoons_{\mathtt{CMC}} t'$.*

Proof. If $s \preceq s'$, then $s' \in \overrightarrow{\mathcal{C}}(\{s\})$ by definition of $\overrightarrow{\mathcal{C}}$ and, since $s \rightleftharpoons_{\mathtt{CMC}} t$ by hypothesis, there is $t' \in \overrightarrow{\mathcal{C}}(\{t\})$ such that $s' \rightleftharpoons_{\mathtt{CMC}} t'$ by Def. 21. Similarly, if $s \succeq s'$, then $s' \in \overleftarrow{\mathcal{C}}(\{s\})$ by definition of $\overleftarrow{\mathcal{C}}$ and, since $s \rightleftharpoons_{\mathtt{CMC}} t$ by hypothesis, there is $t' \in \overleftarrow{\mathcal{C}}(\{t\})$ such that $t \rightleftharpoons_{\mathtt{CMC}} t'$. □

We finally note that CM-bisimilarity and logical equivalence w.r.t. $\mathtt{SLCS}_\eta$ are *incomparable.*

A summary of the relationship between the various equivalences is reported in Figure 5 representing them with their set-inclusion relation as a poset.

5 Conclusions and Future Work

We have introduced $\mathtt{SLCS}_\eta$, its interpretation on face-poset models and the logical equivalence $\equiv_{\mathtt{SLCS}_\eta}$ it induces. We have presented an encoding of $\mathtt{SLCS}_\eta$ into $\mathtt{SLCS}_\gamma$ that we used for proving that $\equiv_{\mathtt{SLCS}_\gamma} \subseteq \equiv_{\mathtt{SLCS}_\eta}$. It is easy to see that $\equiv_{\mathtt{SLCS}_\gamma}$ is strictly stronger than $\equiv_{\mathtt{SLCS}_\eta}$, i.e. $\equiv_{\mathtt{SLCS}_\gamma} \subset \equiv_{\mathtt{SLCS}_\eta}$. We have then compared both equivalences with equivalences proposed in the literature for finite closure models, and in particular CM-bisimilarity, CMC-bisimilarity and CoPa-bisimilarity. It turns out that, for posets CMC-bisimilarity and CoPa-bisimilarity coincide and CMC-bisimilarity is strictly stronger than $\equiv_{\mathtt{SLCS}_\gamma}$ that is strictly stronger

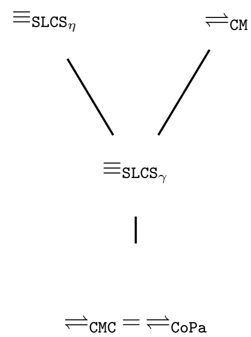


Fig. 5: Hasse diagram of the poset of face-poset model equivalences

than both CM-bisimilarity and $\equiv_{\mathtt{SLCS}_\eta}$, the latter being incomparable. We plan to investigate possible definitions of bisimilarities on face-poset models that coincide with $\equiv_{\mathtt{SLCS}_\eta}$ and possible minimisation algorithms for such bisimilarities. This would represent the best solution for model reduction, that would contribute to improving the performance of model-checking algorithms for $\mathtt{SLCS}_\eta$. We also will investigate approaches for minimisation algorithms for $\equiv_{\mathtt{SLCS}_\gamma}$, or, equivalently for $\pm$-bisimilarity. At the same time, existing efficient minimisation algorithms for CMC-bisimilarity or CoPa-bisimilarity are a good, non optimal solution given the relationship we have proved in this paper between such equivalences and $\equiv_{\mathtt{SLCS}_\gamma}$ and $\equiv_{\mathtt{SLCS}_\eta}$.

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