Note on an extremal problem arising from the diagnosis of regularly interconnected systems

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Abstract- Motivated by the problem of identifying the faulty units in regular interconnected systems this paper studies the combinatorial problem of the following type: for a graph G(V,E) #V=N and any integer □, what is the minimal number f(G,□) such that the removal of f(G, □) nodes results in a graph with a maximal connected component of □ nodes or less.
The graphs that are studied are regular (all nodes have the same degree) or quasi–regular graphs; the main attention is paid to planar and toroidal grids.
The main result is: f(G, □) ≥ N min□ $\frac{I(G,□)-2}{2□ + I(G,□)-2}$, 1□□□□□; I(G,□) is an isoperimetric function: the lower bound on the edges that must be used to connect the nodes that must be removed to isolate a subgraph of □ nodes. Using that inequality we find new and better lower bounds.

Keywords: isoperimetric inequalities, system-level diagnosis, toroidal grids, planar graph.

Categories and subject descriptors: G.2.2 [**Discrete Mathematics**]: *Graph Theory*, B.1.3 [**Control Structures and microprogrammin**g]:Control Structure reliability,Testing and Fault-Tolerance- *Diagnostics*, C.1.2 [**Processor Architectures**]:Multiple Data Stream Architectures- *Array and vector Processors* .

1. INTRODUCTION

The problem that we study has recently engaged some researchers who study the diagnosis of faulty processors in parallel systems with thousands of processors connected in regular interconnected structures.

System-level diagnosis aims at diagnosing systems composed of processors connected by point-to-point, bidirectional links. Every processor is tested by at least another processor which is interconnected with the one undergoing test. Essentially, a test is performed as follows:

- the *testing* processor requests the *tested* processor to run a suitable program;
- the *tested* processor returns an output to the testing processor;
- the *testing* processor compares the received output against an oracle. Every test yields a binary test outcome.

The syndrome is the set of binary outputs of tests. The tests are represented as directed edges in a diagnostic graph where a directed edge (v_iv_j) exists if and only if unit v_i tests unit v_j. It may happen that the interconnection graph and the diagnostic graph are the same.
The last step is a diagnostic algorithm, which has to decode the syndrome and to identify faulty units, working units and the units which cannot be classified as either faulty or good. The diagnosis is complete if either all units are identified faulty or good; on the contrary it is incomplete if at least one unit cannot be classified either faulty or good.

Preparata et al. have introduced the notations of *one-step* and *sequential diagnosis* [1]. In the one-step diagnosis the object is to identify all the faulty units before any faulty unit is repaired or replaced. It is known that in the case of regular interconnected systems a correct and complete diagnosis is possible only when the number of faulty units is no more than a limited and rather small value. The goal of sequential diagnosis is to identify all the faulty units using several diagnosis and repair phases.

A system is said sequentially-t_s-diagnosable if at least one faulty unit can be always identified in the occurrence of an arbitrary set N_f of faulty units with □N_f□□ t_s. The maximum value of t_s is called the sequential diagnosability of the system. Once identified, the faulty units can be repaired or removed and by repeating the diagnosis procedure all the faulty units can be identified.

In [2] a generalized sequential diagnosis algorithm is described (PARTITION algorithm); the degree of diagnosability of this PARTITION algorithm is based on the evaluation of the k-partition number of the diagnostic graph, defined as the largest integer □(G,k) such that the subgraph G' of G induced by N-X, X□□(G,k) contains at least one connected component of cardinality greater than, or equal to k. Assuming t faulty units in the system, if □(G,t+1)>t, then G' must have a connected component of t+1 nodes that must be fault-free.

To estimate □(G,k) the authors find more convenient to use the closely releted function □(G,k) = minimal number m such that removal of m nodes results in a graph with a maximal connected component of k-1 nodes or less; □(G,k)= □(G,k)-1.

The authors find an upper bound S*(G,t) for the vertex degree of a t-nodes-connected subgraph of G; using the convexity of the function S*(G,t) they find an upper bound on the sum of the degrees of the subgraphes induced by $N-X_t$ vertex (X_t =□(G,t+1)):

$$[(N-X_t)/t]\ S^*(G,t)$$

then, if □(G) is the maximum vertex degree, t must satisfy the inequality: $[(N-X_t)/t]\ S^*(G,t)+2\square(G)X_t \geq S^*(G,N)$ that provides a lower bound for t.

They prove that for every nonnegative integer $t \leq N$, N=□V□:

$\square(G_d,t+1) \geq \dfrac{N-(tN^{d-1})^{1/d}}{1+t^{1/d}}$ if G_d is a d-dimensional grid graph defined as Cartesian product of d paths of n vertices, $N=n^d$;

$\square(C_d,t+1) \geq \dfrac{N \log(N/t)}{\log(N^2/t)}$ if C_d is a d-dimensional cube of $N=2^d$ vertices.

In [3,4,5,6,7,8] a diagnostic algorithm (EDARS, Efficient Diagnosis Algorithm for Regular Structures) is presented; for any given syndrome □, EDARS provides the diagnosis if the faulty units are less than a bound t(□) syndrome-dependent that is asserted by the algorithm itself along the diagnosis procedure. EDARS is organized in three steps, called *Local Diagnosis*, *Fault-Free Core Identification* , and *Augmentation*. The first step partitions set N into subsets F (possibly empty of faulty units), D (disjoint pairs with the property that, for every pair, at least one unit is faulty),and Z (Z=N-(F□D)); the second step partitions the subgraph Z in strongly connected components, if □ is the cardinality of the maximum connected component, then □ units are faulty-free if the bound t(□)=□+□F□+□D/2□ of faulty units is satisfied. The last step augments the faulty-free and the faulty units exploiting the reliable tests made by faulty-free units. A syndrome independent bound T(G) that assures the diagnosis in the worst case is evaluated. As T(G)=min(□+D(G,□)/2), the bound is based on the evaluation of the the function D(G,k).

The authors find a lower bound P*(G,□), the boundary of the subgraph, that is the nodes that must be removed to isolate a subgraph of □ nodes of G, then they find (but the proof is rather complicated) D(G, □) as lower bound for the union of the boundary of all the subgraphs (of at most □ nodes) of G.

They prove that for every nonnegative integer T(G)□N/2, N=□V□= n^2 , D(G,□) ≥2(T(G)-□):

$D(G_{2\text{-}4},\square) \geq \dfrac{N((17/18)\ \overline{2\square\text{-}1}\ \text{-}1/6)}{\square+(17/18)\ \overline{2\square\text{-}1}\ \text{-}1/6}$; $G_{2\text{-}4}$ is a 2-dimensional toroidal grid defined as Cartesian product of 2 cicles of n vertices,

$D(\underline{G}_{2\text{-}4},\square) \geq \dfrac{(n+2n\text{-}4/3)[(17/18)\ \overline{2\square\text{-}1}\ \text{-}1/6]}{\square+(17/18)\ \overline{2\square\text{-}1}\ \text{-}1/6}$ -2n+4/3; $\underline{G}_{2\text{-}4}$ is a 2-dimensional planar grid defined as Cartesian product of 2 paths of n vertices,

$D(\underline{G}_{2\text{-}6},\square) \geq \dfrac{(n^2+2n)(1+\ \overline{12\square\text{-}3})}{2\square+1+\ \overline{12\square\text{-}3}}$ -2n; $\underline{G}_{2\text{-}6}$ is a $D(\underline{G}_{2\text{-}4},\square)$ planar grid with one diagonal in each cell,

$D(\underline{G}_{2\text{-}8},\square) \geq \dfrac{(n+1)^2(1+2\ \ \square)}{\square+1+2\ \ \square}$ -2n-1; $\underline{G}_{2\text{-}8}$ is a $D(\underline{G}_{2\text{-}4},\square)$ planar grid with two diagonals in each cell,

$D(\underline{G}_{2\text{-}3},\square) \geq \dfrac{(n^2+n)\ \overline{6\square}}{3\square+\ \overline{6\square}}$ -n; $\underline{G}_{2\text{-}3}$ is obtained deleting n/2 edges in alternate positions of every row of a $D(\underline{G}_{2\text{-}4},\square)$ planar grid.

The partitioning problem and the problem of finding a lower bound for the union of the boundary are the same following problem:

for a graph G(V,E), □V□=N and any integer □, what is the minimal number f(G, □) such that removal of f(G, □) nodes results in a graph with a maximal connected component of □ nodes or less.

The same problem has been considered also in [9] for a d-dimensional cube C_d ($N=2^d$) motivated by the need of guarantee that a quorum of □>N/2 processors are connected, condition that may be necessary to assure a correct working of a system of processes operating asynchronously in unreliable networks.

The authors find two relations for $f(C_d,\square)$ using Harper's isoperimetric theorems:

$f(C_d,\square) \geq 2^{d-1}(d-k)/d$ for k, 0□k□d-1, if 2^k □□<(4/3) 2^k

$f(C_d,\square) \geq 2^d(2k-d)/(d^2-dk+2k)$ if k>d/2 is max-number such that □□ (d!d)+.. (d!k)-1

The first relation gives better results and is obtained by the elementary inequality : f(G, □) ≥ □(G, □)/d where □(G, □) is the maximal number such that the removal of □(G, □) edges results in a graph with a maximal connected component of A nodes or less.
The second relation is obtained finding P*(C_d,□) and using the inequality f(G, □) ≥(□ $_i$ P*(C_d,□ $_i$))/d

Our approach uses the Euler characteristic equation for polyhedra of genus □□1 and an isoperimetric relation: the lower bound on the edges that must be used to connect the nodes that must be removed to isolate a subgraph of □ nodes.
The paper is structured as follow. In section 2 planar and toroidal graphs are studied and a lower bound for general regular toroidal grids is proved. Section 3 is devoted to 2-dimensional toroidal grids defined as Cartesian product of 2 cicles of n vertices ; quasi-regular planar grids are considered in section 4.

2. Planar and toroidal graphs

Let G(V,E) be a connected graph V=□v_1,v_2,… v_N□, E=□e_1,e_2,… $e_{□E□}$□ (no loops or multiple edges are permitted), and let S=□ S_1, S_2,… $S_{□S□}$□, □S□=s, be the set of connected subgraphs of at most □ nodes obtained by the removal of the nodes of a set F(G,□) .

Let are S_i =S_i (V_i,E_i), V_i□V and E_i□E; then □S_i□□□,□i, 1□ i □ s.
Denote by P(S_i)=□S_i-S_i the collection of all nodes that are removed to isolate the subgraph S_i, then:
F(G,□) =F = □ $_i^s$ P(S_i) and V= F(G,□)+(□ $_i^s$ V_i). Let be f(G,□)= □F(G,□) □.

Lemma1: Let S=□ S_1, S_2,… $S_{□S□}$□, □S□=s, be the set of connected subgraphs of at most □ nodes obtained by the removal of the nodes of a set F(G,□); if v_i□F(G,□) and it does not exist v_k□S_i, S_i□S, v_iv_k□E then F(G,□) is reducible.
In fact also the set F'(G,□)= F(G,□)-v_i is such that the removal of the nodes of F'(G,□) disconnects the graph in S'=□ S_1, S_2,… $S_{□S□}$, v_i □, connected subgraphs of at most □ nodes.

Remark 1. As we are interested to know the minimum value of F(G,□) we will suppose that F(G,□) is not a reducible set, and such that v_i□F(G, □) only if □S_i□S, v_k□S_i , v_iv_k□E as the Lemma 1 asserts.

Remark 2. Some nodes can be removed not to disconnect a component from some other component, but to reduce the cardinality of a component. It does not prevent us to obtain a minimum set if we suppose in the following that these nodes are removed on the boundary of the component.

In Fig.1a a planar graph of 24 nodes is drown, 4 components S_1,S_2, S_3,S_4 of at most 4 nodes are isolated by the removal of the 15 black marked nodes.

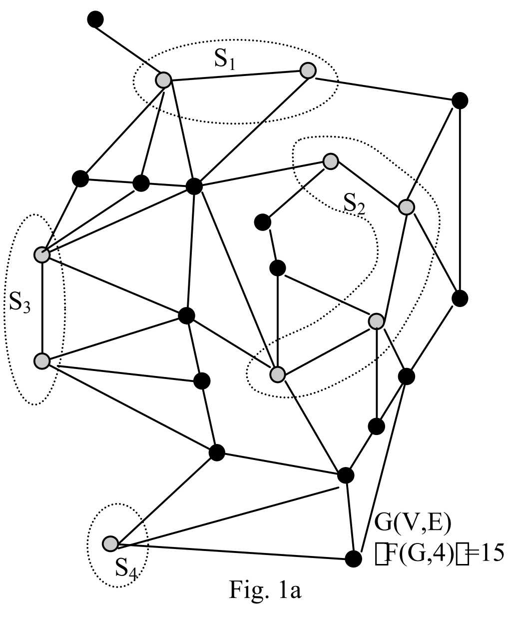


Fig. 1a

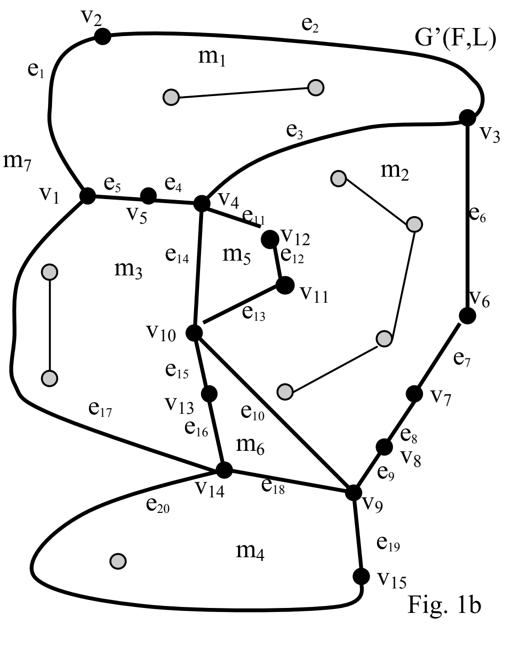


Fig. 1b

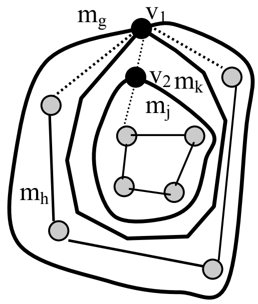


Fig. 1c

s=4, m=7
□F□=□v_1,.. ,v_{15}□
L=□e_1, .. ,e_L□= 20
l_1=5, l_2=9, l_3=6, l_4=3,
l_5=4, l_6=4, l_7=9

Let be □ the orientable surface of genus □<2 on which the graph G is embedded [10].

Let $G'(F,L)$ be the pseudograph that can be obtained connecting, for every component S_i, all the nodes of $P(S_i)$ in such a way that every connected component S_i is contained in a one's own region of the surface Σ by means of one or more boundary cycles; let m_i be the region that contains S_i.
$G'(F,L)$ can have loops (Fig. 1c).
The surface Σ is partitioned in $m \geq s$ regions $m_1, m_2, \ldots m_s, m_{s+1}, \ldots m_m$; s are the regions that contain the s components, $m-s \geq 0$ (at least one contains the infinite region of the plan if G is a planar graph) are empty. Fig.1b shows the graph $G'(F,L)$ obtained from the planar graph ($\gamma=0$) of Fig. 1a that defines m=7 regions by $|F|=15$ vertices and $|L|=20$ edges.

Lemma2: If we exclude the infinite region of the plan (if G' is a planar graph), every empty region is bounded by 2 edges at least.

In fact: the infinite region of the plan, that must exist if G' is a planar graph, can border on only one region and by only one edge (a loop); every other empty region is created by different edges belonging to boundary cycles of 2 not empty regions at least and if an empty region separates 2 regions at least 2 edges must exist that connect the same 2 nodes or are 2 loops.

The graph $G'(F,L)$ can be composed by more disconnected components. We will call the graph $G'(F,L)$ the "*boundary graph*" of G induced by the set $F(G,k)$.

As $G'(F,L)$ is the union of cycles that bound not overlapped regions of the surface Σ on which $G(V,E)$ is embedded, then:

Lemma3: if $G(V,E)$ is embedded on the surface Σ of genus $\gamma(G)$, the boundary graph *$G'(F,L')$* is embedded on a surface Σ' of genus $\gamma(G') \leq \gamma(G)$.

The graph $G'(F,L)$ of genus $\gamma(G')$ on the surface Σ represents a polyhedron of $|F|$ vertices, $|L|$ edges, $m \geq s$ faces, and for the Euler equation generalized to polyhedra of arbitrary genus:

$$|F|+m-|L| \geq 2-2\gamma \qquad 1)$$

Let l_i and n_i be respectively the number of edges and the number of nodes on the boundary cycles that bound region i-th S_i , then:

$$l_i \geq n_i=|P(S_i)| \qquad 2)$$

It may happen that $l_i \geq n_i$ as a node can belong many times to the same cycle (Fig. 1c,2a,2b).
Suppose now that G' is embedded on a torus, $\gamma(G') \leq 1$, and then represented as a rectangle in which both pairs of opposite sides are identified; as every edge is the boundary of two regions, then:

$$\sum^{m}_{i=1} n_i \leq \sum^{m}_{i=1} l_i = 2|L|$$

$$|L| = \sum^{m}_{i=1} l_i /2 \qquad 3)$$

replacing 3) in 1):

if $\gamma(G')=0$: $\quad |F|+m - \sum^{m}_{i} (l_i/2) \geq 2$

$$|F| \geq 2+ \sum^{m}_{i=1} (l_i-2)/2 = 2+ \sum^{s}_{i=1} (l_i-2)/2+ \sum^{m}_{i=s+1} (l_i-2)/2$$

for Lemma2 : $\quad |F| \geq \sum^{s}_{i=1} (l_i-2)/2+1.5 \qquad 4)$

if $\gamma(G')=1$: $\quad |F|+m - \sum^{m}_{i} (l_i/2) \geq 0$

$$|F| \geq \sum^{m}_{i=1} (l_i-2)/2 \geq \sum^{s}_{i=1} (l_i-2)/2 \qquad 4')$$

if $\gamma(G') \leq 1$ $\quad |F| \geq \sum^{m}_{i=1} (l_i-2)/2 \geq \sum^{s}_{i=1} (l_i-2)/2 \qquad 5)$

and for 2) : $\quad |F| \geq \sum^{m}_{i=1} (n_i-2)/2 \geq \sum^{s}_{i=1} (n_i-2)/2 \qquad 5')$

In conclusion it is possible to state that if a graph $G(V,E)$ is disconnected in s components removing vertices that must be removed and if the boundary graph G' is embedded either on a plan or on a torus, then the number of vertices removed is at least the sum of half-1 of the number of edges that bound every region. If $n_i = l_i$, every region (even if empty) contributes on an average of half-1 of the number of nodes on his boundary at least.
The 5) and 5') can be used to evaluate a lower bound for $f(G,k)$ if the boundary graph of G is embedded on a torus.
Let $l(G,k) \geq 2$ be an isoperimetric function for the minimum number of edges that bound the region that contains a connected component S_i of k nodes connecting all the $P(S_i)$ nodes and
let $n(G,k) \geq 2$ be an isoperimetric function for the minimum number of nodes that must be deleted to disconnect a component S_i of k nodes, $P(S_i) \geq n(G,k)$, then:

$$l_i \geq l(G,k_i) \geq n(G,k)$$

If all the s subgraphs have □ nodes from 5):

f(G,□) ≥ ^{m}□$_{i=1}$ (l$_i$ –2)/2 ≥ m(l(G,□)–2)/2) ≥ s(l(G,□)–2)/2

N= s□ + f(G,□) ≥ s□+ s(l(G,□)–2)/2

Let be □(G,□) the function density so defined:

□(G,□) =((l(G,□)–2)/2)/(□+((l(G,□)–2)/2)= (l(G,□)–2)/(2□ +(l(G,□)–2)) 6)

then: f(G,□)/N≥ (s(l(G,□)–2)/2)/(s□+ s(l(G,□)–2)/2))

f(G,□)/N≥ □(G,□)

It is easy now to prove the following Theorem:

Theorem 1. *Let G(V,E) be a graph, let be f(G,□) such that removal of f(G,□) nodes results in a graph with a maximal connected component of at most □ nodes, then, if the boundary graph G'(F,L') is a toroidal graph (□(G')□1) and if l(G,□) ≥ n(G,□) ≥2, 1□ □□ □ :*

f(G,□) ≥ N min□ $\frac{l(G,□) - 2}{2□ + l(G,□) - 2}$, 1□ □□ □□ 7)

proof : from 5): f(G,□) ≥ ^{m}□$_{i=1}$ (l$_i$–2)/2 ; f(G, □) =□+ ^{m}□$_{i=1}$ (l$_i$–2)/2 =□+ ^{s}□$_{i=1}$ (l$_i$–2)/2+ ^{m}□$_{i=s+1}$ (l$_i$–2)/2 , □≥0

as: N= ^{m}□$_{i=1}$ □$_i$ + f(G,□) = ^{s}□$_{i=1}$ □$_i$+□+ ^{s}□$_{i=1}$ (l$_i$–2)/2+ ^{m}□$_{i=s+1}$ (l$_i$–2)/2

f(G,□)/N=(□+ ^{s}□$_{i=1}$ (l$_i$–2)/2+ ^{m}□$_{i=s+1}$ (l$_i$–2)/2)/(□+^{m}□$_{i=1}$ □$_i$+ ^{s}□$_{i=1}$ (l$_i$–2)/2+ ^{m}□$_{i=s+1}$ (l$_i$–2)/2)

as f(G,□)/N<1, and □+^{m}□$_{i=s+1}$ (l$_i$–2)/2 ≥0 (for a planar graph □≥2 and ^{m}□$_{i=s+1}$ (l$_i$–2)/2 ≥-1/2), then:

f(G,□)/N ≥ (^{s}□$_i$ (l$_i$–2)/2)/(^{s} □$_i$ □$_i$+ ^{s}□$_i$ (l$_i$–2)/2)= ^{s}□$_i$ (l$_i$–2)/(^{s} □$_i$ 2□$_i$+ ^{s}□$_i$ (l$_i$–2)) 7')

as l$_i$ ≥ l(G,□ $_i$), from 6): □(G,□$_i$)= ((l(G,□ $_i$)–2))/(2□$_i$+(l(G,□ $_i$)–2)) ≤ (l$_i$ –2)/(2□$_i$+(l$_i$ –2))

then : (l$_i$–2)≥□(G,□$_i$) (2□$_i$+(l$_i$–2))

replacing in 7'): f(G,□)/N≥ ^{s}□$_i$ □(G,□$_i$) (2□$_i$+(l$_i$–2))/(^{s}□$_i$ 2□$_i$+ □$_i$(l$_i$ –2))

if □(G,□$_x$) = min□□(G,□$_i$) □□$_i$, i=1….s□= min□(l(G,□)–2)/(2 □+l(G,□)–2), 1□□□□□

f(G,□)/N ≥ □(G,□$_x$) □$_i$ (2□$_i$+(l$_i$–2))/(^{s}□$_i$ 2□$_i$ + ^{s}□$_i$ (l$_i$ –2))=□(G,□$_x$)

□

Corollary 1. If □'(□)□□(□) is a monotonically decreasing function in □, then:

f(G,□) ≥ N□(□) 8)

In next sections we will apply Theorem 1 and Corollary 1 to some regular or quasi regular structures *G(V,E)* with l(G,□) >2 and with toroidal boundary graphs *G'(F,L')*.

3. two-dimensional toroidal grids of nodes of degree 4

Let $G_{2,4}(V,E)$ be the 2-dimensional grid toroidal graph of $N=n^2$ nodes of degree 4 defined as Cartesian product of 2 loops of n nodes (Fig.2):

if □□□(n-1)^2/2□ f($G_{2,4}$,□) ≥ N □$_{min}$(□) = N min□ $\frac{□2 \; \overline{2□-1}□}{2□+□2 \; \overline{2□-1}□}$,□□□□□ 9)

if □>□(n-1) ^{2}/2□ f($G_{2,4}$,□) ≥ min[2n-2 , □2+2 (2□-1)□]; □=□N-□+1- (8(N-□+1)-20)□.

Proof : for a graph G $_{2,4}$(V,E) [4, 6]: l(G $_{2,4}$,□)=n(G $_{2,4}$,□)=□2+2 (2□-1)□ where G $_{2,4}$(V,E) is an infinite grid defined as Cartesian product of 2 paths of infinite nodes in the plan.

It is easy to verify that if 2+□2 (2□-1)□<2n, □<□(n-1) ^{2}/2□and that in the toroidal grid $G_{2,4}$(V,E) l($G_{2,4}$,□) □2n.

Let be n even: removing 2n-2 nodes, the grid $G_{2,4}$(V,E) (Fig.2a) can be partitioned in two components of

□$_1$=(n^2-2n+2)/2= [(n-1)^2+1]/2 = □(n-1)^2/2□,

l($G_{2,4}$, □$_1$) =2n= 2+ □2 (2□$_1$-1)□, f($G_{2,4}$,□$_1$)$_{min}$ =2n-2

and □$_{min}$(□$_1$)=(2n-2)/N=(2n-2)/(2□$_1$+2n-2)= □2 (2□$_1$-1)□/ (2□$_1$+□2 (2□$_1$-1)□).

Let be n odd: removing 2n-1 nodes, the grid (Fig.2b) can be partitioned in 2 components of

□$_2$=(n-1)^2/2 =□(n-1)^2/2□, l($G_{2,4}$,□$_2$) = 2n, f($G_{2,4}$,□$_1$)$_{min}$ =2n-2

□$_{min}$(□$_2$)=(2n-1)/N>□ $_{min}$(□$_1$).

Then if □=□(n-1)^2/2□, □ $_{min}$(□)□□2 (2□$_1$-1)□/(2□$_1$+□2 (2□$_1$-1)□) and □+N□ $_{min}$(□)□□+2n-2=((n+1)^2/2)-3/2

As if we delete from $G_{2,4}$(V,E) a toroidal subgraph of □ nodes the remaining graph is embedded on a plan, then l($G_{2,4}$,□)= l(G $_{2,4}$,□)= n(G $_{2,4}$,□)= □2+2 (2□-1)□ where □ must satisfy the relation: □+l(G $_{2,4}$,□)= n^2-□ .

If □<□(n-1) ^{2}/2□, □+l(G $_{2,4}$,□)>((n+1)^2/2)-1, □>□ and l($G_{2,4}$,□)=l(G $_{2,4}$,□)> l(G $_{2,4}$,□).

Then if □□□(n-1) ^{2}/2□, $I(G_{2,4},□)$ = $I(G_{2,4},□)$ =□2+2 (2□-1)□, □(□)=□2 (2$□_1$-1)□/(2$□_1$+□2 (2$□_1$-1)□),
and as the boundary graph $G'_{2,4}$ (F,E') is a toroidal graph, as $I(G_{2,4},□)$ >2 , as □(□) is not a monotonically decreasing function in □, from Theorem 1:

$f(G_{2,4},□) \geq N□_{min}(□) = n^2$ min□(□2 (2□-1)□)/(2□ +(□2 (2□-1)□)) □□□□□ 9')

If □>□(n-1) ^{2}/2□ the graph can be partitioned in 2 components □ and □<□ such that:

□+ $n(G_{2,4},□) \geq n^2$ + □

□+□2+2 (2□-1)□≥ n^2 + □

□=□N-□+1- (8(N-□+1)-20)□

as a component of □ nodes must be disconnected removing at least □2+2 (2□-1)□□2n-2 nodes, then:

$f(G_{2,4},□) \geq$ min[2n-2, □2+2 (2□-1)□ , □=□N-□+1- (8(N-□+1)-20)□.

□

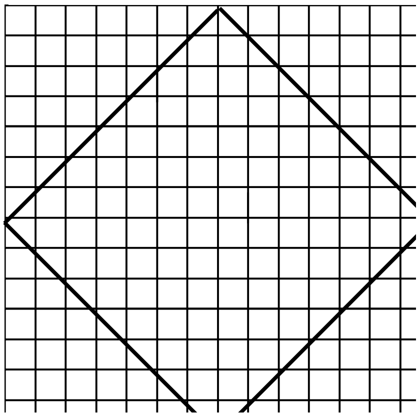


Fig. 2a

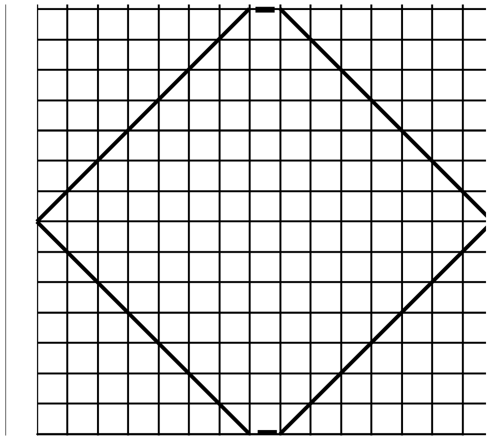
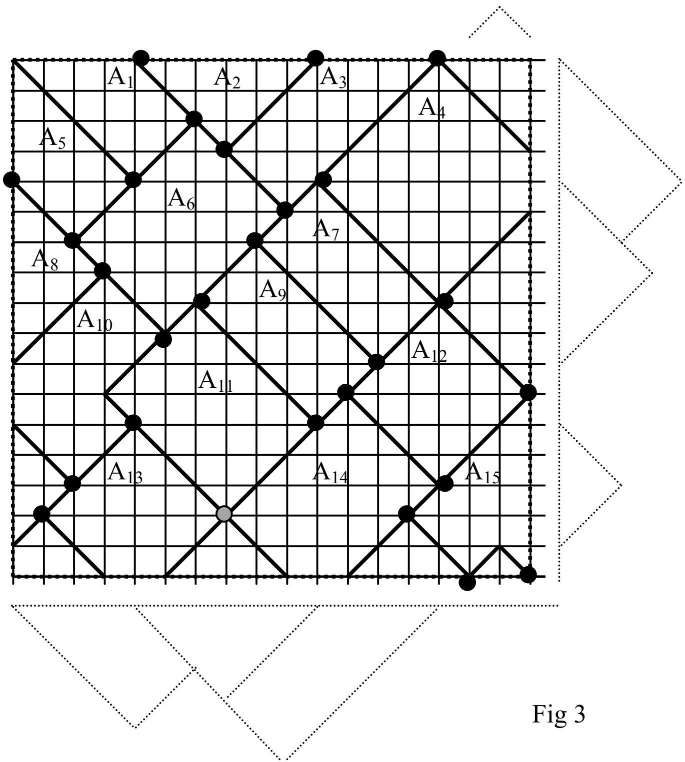


Fig. 2b



N=18x18=324
□F□=88

A_1 =14 , n_1=14 , (n_1-2)/2= 6
A_2 =13 , n_2=12 , (n_2-2)/2= 5
A_3 =17 , n_3=16 , (n_3-2)/2= 7
A_4 =25 , n_4=16 , (n_4-2)/2= 7
A_5 =17 , n_5=16 , (n_5-2)/2= 7
A_6 =18 , n_6=14 , (n_6-2)/2= 6
A_7 =11 , n_7=12 , (n_7-2)/2= 5
A_8 =18 , n_8=14 , (n_8-2)/2= 6
A_9 =11 , n_9=12 , (n_9-2)/2= 5
A_{10} =14 , n_{10}=14 , (n_{10}-2)/2=6
A_{11} =18 , n_{11}=14 , (n_{11}-2)/2=6
A_{12} =13 , n_{12}=12 , (n_{12}-2)/2=5
A_{13} =13 , n_{13}=12 , (n_{13}-2)/2=5
A_{14} =18 , n_{14}=14 , (n_{14}-2)/2=6
A_{15} =16 , n_{15}=14 , (n_{15}-2)/2=6

$□_i$ (n_i –2)/2= 88

$f(G_{2,4},25)$=324x7/32=70,875

Fig 3

Fig. 3 shows an example were a toroidal grid of 18^2 nodes is partitioned in 15 subgraphs deleting 88 nodes and the 5) is verified with the minimum value.

□(□) is not a monotonically decreasing function in □, indeed:
□=24 ; □≥ √(2□-1) □=14 ; □(24)≥14/(48+14)=7/31=0.22580…
□=25 ; □≥ √(2□-1) □=14 ; □(25)≥14/(50+14)=7/32=0.21875 <□(24)
□=26 ; □≥ √(2□-1) □=15 ; □(26)≥15/(52+15)=15/67=0.22388… >□(25)
□=27 ; □≥ √(2□-1) □=15 ; □(27)≥15/(54+15)=15/69=0.21739… <□(25)
□=28 ; □≥ √(2□-1) □=15 ; □(28)≥15/(56+15)=15/71=0.21126… <□(27)
than: $f(G_{2,4},25) \geq N$ □(25) as □(25)< □(B) □B<25
$f(G_{2,4},26) \geq N$ □(25) as □(25)□ □(B) □B□26
$f(G_{2,4},27) \geq N$ □(27) as □(27)< □(B) □B□27

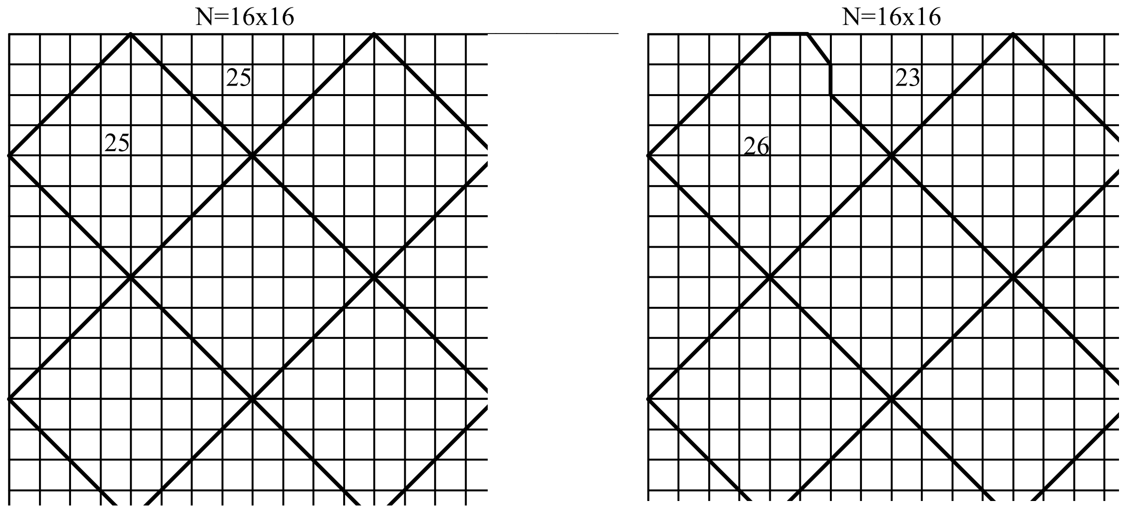


Fig 4a Fig 4b

In conclusion if in a grid of $N=16^2$ nodes it is possible to delete N□(25)=256x0.21875=56=8x7 nodes to obtain 8 subgraphs of 25 nodes (Fig.4a), at least the same number of nodes must be deleted (may be 1 more) if the greatest subgraph has 26 nodes (Fig.4b).

It is easy to verify that there are values of □ and N that satisfy the 9) with the minimum value.
Fig.5a shows a grid 17x17 partitioned in 17 graphs of 12 nodes removing 17^2-(17x12)=85 nodes and from 9): $f(G_{2,4},□)_{min} \geq$ □(12)17^2=(5/17) 17^2.
Fig.5b shows a grid 22x22 partitioned in 22 graphs of 16 nodes removing 22x6=132 nodes; this partition does not satisfy 9) with the minimum value as f(16)≥ □(16)22^2=(11/41)22^2=130.
Fig.5c shows a grid 9x9 partitioned in 6 graphs of 9 nodes removing 81/3=27 nodes.

We can obtain a function more approximate than 9); as: □(□)= √(2□-1)/[□+ √(2□-1)] □ □(□) is a monotonically decreasing function, from 8):

$$f(G_{2,4},□) \geq N \frac{\sqrt{2□-1}}{□ + \sqrt{2□-1}} \quad ; \quad □□ □(n-1)^2/2□ \qquad 10)$$

The 9) and 10) give a better bound than that in [4].

Using the Khanna-Fuchs approach (or the edge isoperimetric inequality derived in [11]), if G_d is a toroidal d-dimensional grid graph defined as Cartesian product of d loops of n vertices, $N=n^d$, as the number of edges is 2dN , □X□□= $f(G_d,□)_{min}$ must satisfy the following inequality:

$$2d(□-□^{d-1/d})(N-□X_□□)(1/□)+4d□X_□□\geq 2dN$$

from which we get the following lower bound: $□X_□□ \geq N \frac{□^{(d-1)/d}}{□+□^{(d-1)/d}}$

and for d=2: $□X_□□ \geq N \frac{□}{□+□}$

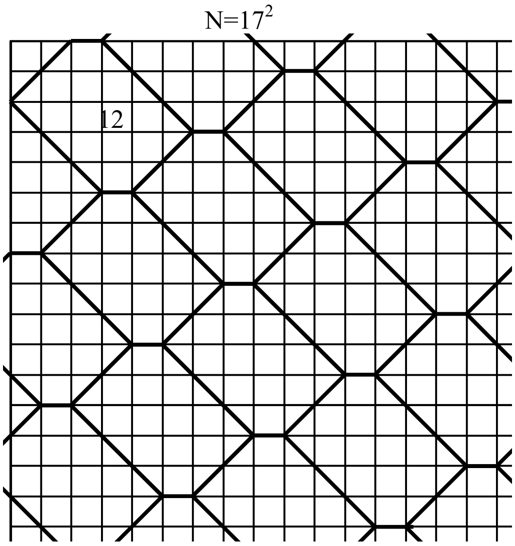


Fig 5a

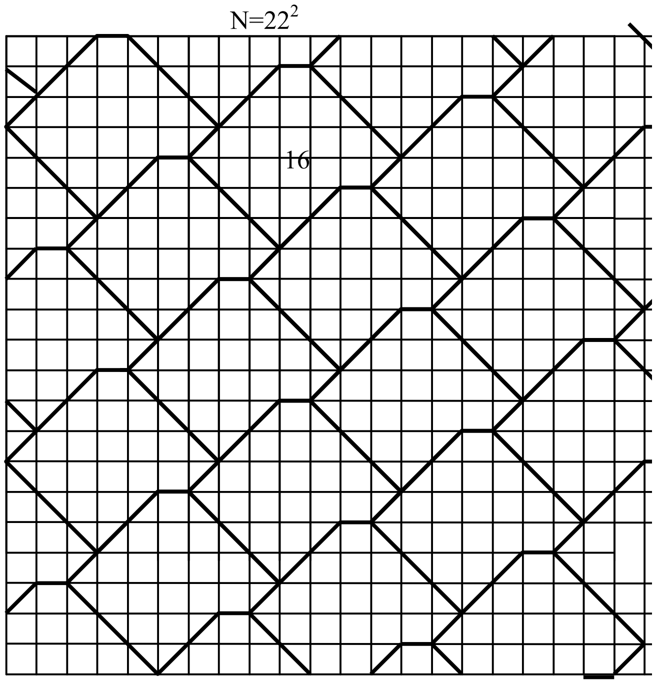


Fig. 5b

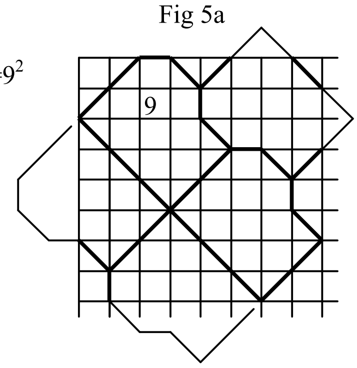


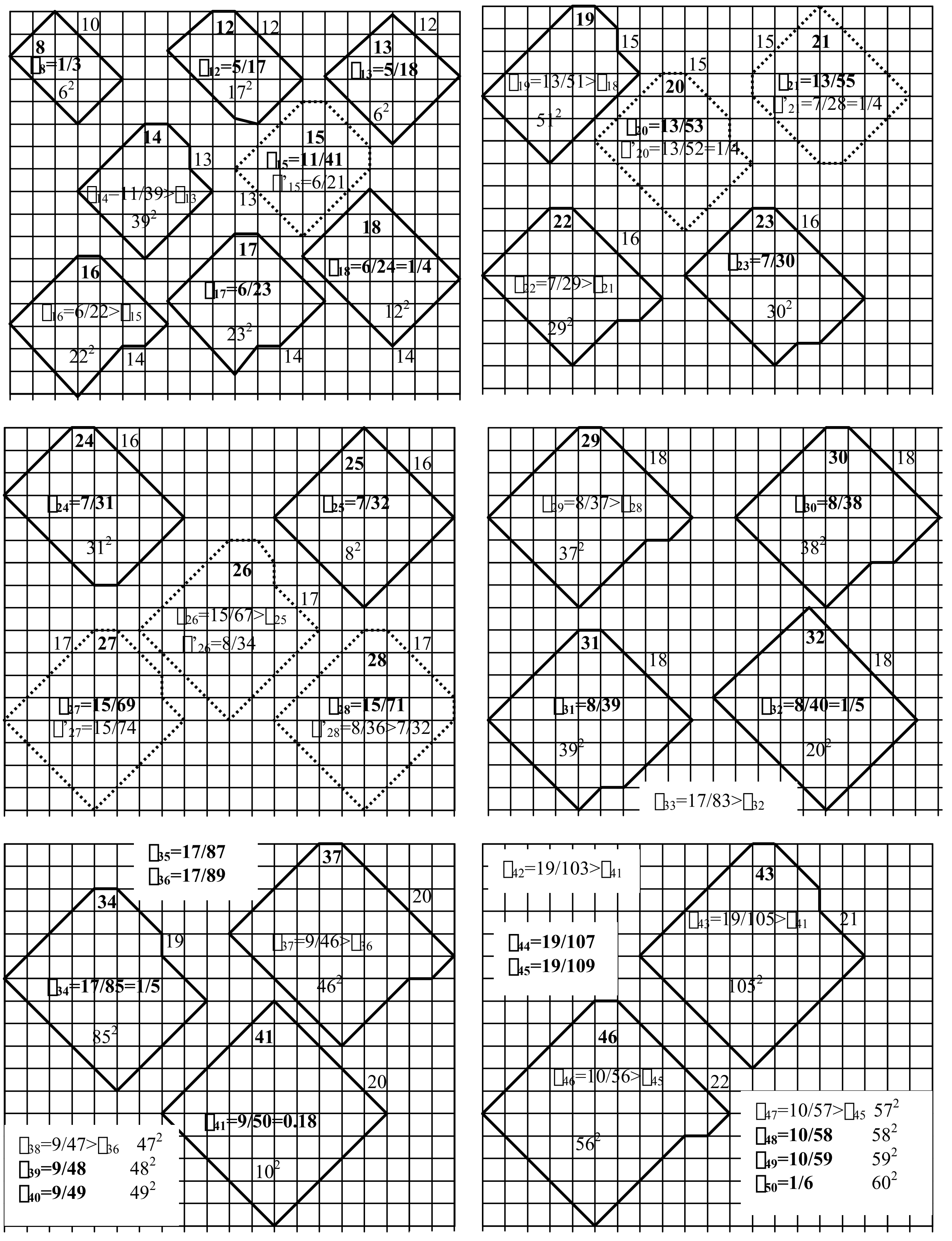
Fig. 5c

□	□(□)	□(□))	□min((□))	n	n²
8	1/3	0,333333	1/3	6	36
9	1/3	0,333333	1/3	9	81
10	9/29	0,310345	9/29		
11	5/16	0,312500	5/16		
12	5/17	0,294118	5/17	17	289
13	5/18	0,277778	5/18	6	36
14	11/39	0,282051	5/18		
15	11/41	0,268293	11/41		
16	3/11	0,272727	11/41		
17	6/23	0,260870	6/23	23	529
18	1/4	0,250000	1/4	12	144
19	13/51	0,254902	1/4		
20	13/53	0,245283	13/53		
21	13/55	0,236364	13/55		
22	7/29	0,241379	13/55		
23	7/30	0,233333	7/30	30	900
24	7/31	0,225806	7/31	31	961
25	7/32	0,218750	7/32	8	64
26	15/67	0,223881	7/32		
27	5/23	0,217391	5/23		
28	15/71	0,211268	15/71		
29	8/37	0,216216	15/71		

A	□(□))	□(□)))	□min((□))	n	n²
30	4/19	0,210526	4/19	38	1444
31	8/39	0,205128	8/39	39	1521
32	1/5	0,200000	1/5	20	400
33	17/83	0,204819	1/5		
34	1/5	0,200000	1/5	85	7225
35	17/87	0,195402	17/87		
36	17/89	0,191011	17/89		
37	9/46	0,195652	17/89		
38	9/47	0,191489	9/47		
39	3/16	0,187500	3/16	48	2304
40	9/49	0,183673	9/49	49	2401
41	9/50	0,180000	9/50	10	100
42	12/65	0,184466	9/50		
43	2/11	0,180952	9/50		
44	11/62	0,177570	11/62		
45	15/86	0,174312	15/86		
46	5/28	0,178571	15/86		
47	10/57	0,175439	15/86		
48	5/29	0,172414	5/29	58	3364
49	10/59	0,169492	10/59	59	3481
50	1/6	0,166667	1/6	60	3600

Fig. 6a

Fig. 6a tabulates some values of □, □(□)), □min((□)) , and values of n that allow optimal partitions using shapes of Fig. 6b.



□8=1/3
□12=5/17
□13=5/18
□14=11/39>□13
□15=11/41
□'15=6/21
□16=6/22>□15
□17=6/23
□18=6/24=1/4
□19=13/51>□18
□20=13/53
□'20=13/52=1/4
□21=13/55
□'2=7/28=1/4
□22=7/29>□21
□23=7/30
□24=7/31
□25=7/32
□26=15/67>□25
□'26=8/34
□27=15/69
□'27=15/74
□28=15/71
□'28=8/36>7/32
□29=8/37>□28
□30=8/38
□31=8/39
□32=8/40=1/5
□33=17/83>□32
□34=17/85=1/5
□35=17/87
□36=17/89
□37=9/46>□36
□38=9/47>□36 47²
□39=9/48 48²
□40=9/49 49²
□41=9/50=0.18
□42=19/103>□41
□43=19/105>□41
□44=19/107
□45=19/109
□46=10/56>□45
□47=10/57>□45 57²
□48=10/58 58²
□49=10/59 59²
□50=1/6 60²

Fig. 6b

3.1 two-dimensional toroidal grid of nodes of degree 6

Let $G_{2\text{-}6}(V,E)$ be a 2-dimensional toroidal grid of $N=n^2$ nodes of degree 6, n≥3, obtained from $G_{2,4}(V,E)$ with one diagonal in each cell (Fig. 7a), then:

if □□□(n²-2n)/2□ f(G₂,₆,□) ≥ N min□ (1+□ 12□-3□) / (2□+1+□ 12□-3□) , (n-1)/(□+n-1) □□□□□ 11)

if □ >□(n²-2n)/2□ f(G₂,₆,□) ≥ min[2n, 3+□ 12□-3 □ ; □=n²-□+2- 12(n²-□+2)-36

Proof :

In G 2-6 (V,E), I(G 2-6, □) = n(G 2-6, □) =3+□ (12□-3) □ [7, 6] .

It is easy to verify that if 3+□ (12□-3)□<2n, a<(n²-2n)/2 and that in a toroidal grid G2-6 (V,E) (Fig.7b,c)

I(G2-6, □) □ 2n+1.

As the toroidal grid G2-6(V,E), if n is even, can be partitioned in two components of □1=(n²-2n)/2 =□(n²-2n)/2□ nodes by the removal of 2n nodes connected in two loops of n edges; I(G2-6, □1)=2n;

as if we delete from G2,6(V,E) a toroidal subgraph of □ nodes the remaining graph is embedded on a plan, then I(G2,6,□)= I(G 2,6,□)= 3+□ (12□-3)□ where □ must satisfy the relation: □+ I(G 2,6,□)= n²-□ .

But □=□ if □+I(G 2,6, □)=n²-□, or n²-2□- I(G 2,6, □)=0;

□<□ if: n²-2□- I(G 2,6, □)<0, n²-2□- 3-□ (12□-3)□<0 , n²-2□- 3<□ (12□-3)□, n²-2□- 3< (12□-3)

If: n⁴+4□²+9-4n²□-6n²+12□<12□-3 and n²-2□- 3≥0, n²-3≥2□

If: 4□²-4n²□+n⁴ -6n²+12<0, □>[n²- (6n²-12)]/2

But as if 3+□ (12□-3)□<2n, □<(n²-2n)/2 <[n²- (6n²-12)]/2 , (n>1) then □>□ .

Then if 3+□ (12□-3)□<2n, □□□n²/3-4n/3+19/12□-1:(1) I(G2,6,□)=I(G 2,6, □)≥ I(G 2,6,□)= 3+□ (12□-3)□

and if 3+□ (12□-3)□≥2n and □□□(n²-2n)/2□ I(G 2,6, □) ≥2n.

Then if □□□(n²-2n)/2□from Theorem1 :

□(□)= min[(1+□12□-3□)/(2□+1+□12□-3□) , (n-1)/(□+n-1) , □□□□].

If □>□(n²-2n)/2□ the graph can be partitioned in two components □ and □<□ such that:

n² ≥ □ + □+ n(G 2,6,□)

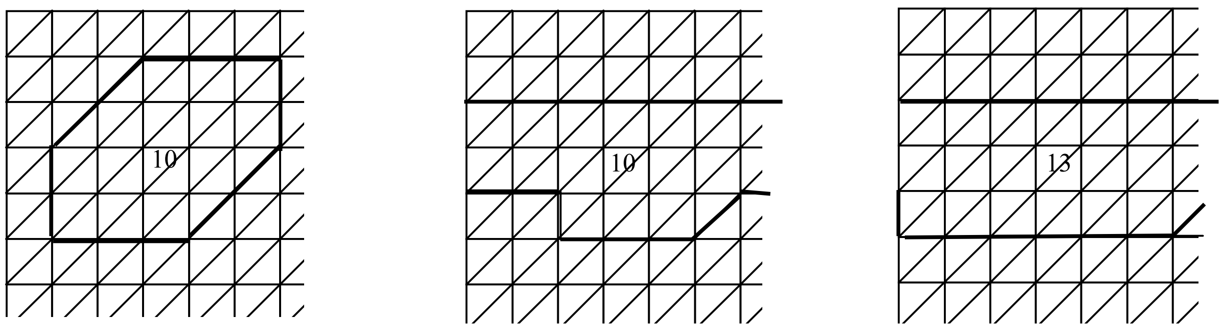


Fig 7a Fig 7b Fig 7c

as a component of □ nodes must be disconnected by removing at least 3+□ (12□-3)□□2n nodes, then:

if □ is the minimum value such that : □+3+□ (12□-3)□≥n²-□

□=n²-□+2- (12(n²-□+2)-36)

f(G2,6,□) ≥ min[2n, 3+□ (12□-3)□ .

□

---

(1) 3+□ (12□-3)□<2n if □ (12□-3)□<2n-3, (12□-3)<2n-4, (12□-3)<(2n-4)², □<n²/3-4n/3+19/12, □□□n²/3-4n/3+19/12□-1

if □≥□n²/3-4n/3+19/12□ then 3+□ (12□-3)□≥3+□ (12□n²/3-4n/3+19/12□-3)□=3+□ (12□n²/3-4n/3+1+7/12□-3)□

if n=(3k+d) then: 3+□ (12□n²/3-4n/3+1+7/12□-3)□= 3+□ (12□(3k+d) ²/3-4(3k+d) /3+1+7/12□-3)□=

=3+□ (12□(9k²+d²+6kd) /3-4k-4d/3 +1+7/12□-3)□= 3+□ (12□3k²+ d²/3+2dk-4k-4d/3 +1+7/12□-3)□=

=3+□ (12□3k²-2k(2-d)+(d²-4d+3)/3+7/12□-3)□=

if d=0 : = 3+□ (12□3k²-4k+ 1+7/12□-3)□= 3+□ (36K²-48K+24 -3)□= 3+□ ((6K-4)²+5)□= 3+(6K-4)+1= 6K= 2n

if d=1 : =3+□ (12□3k²-2k+7/12□-3)□= 3+□ (36k²-24k+9)□= 3+□ ((6k-2)²+5)□> 3+6k-2+1= 6k+2= 2n

if d=2 : =3+□ (12□3k²-1/3+7/12□-3)□= 3+□ (12□3k²+3/12□-3)□= 3+□ (12(3k²+1) -3)□=3+□ (36k²+9)□=4+6k=2n

Let be: $□'(□)= \frac{1+\sqrt{12□-3}}{2□+1+\sqrt{12□-3}}$ □ □(□) , the 11) , if □□ □$(n^2-2n)/2$□, can be approximated by :

$$f(G_{2\text{-}6},□) \geq N□'(□)=N \min\left[\frac{1+\sqrt{12□-3}}{2□+1+\sqrt{12□-3}} , \frac{n-1}{□+n-1} \right] \quad 12)$$

3.2 two-dimensional grid of nodes of degree 8

As 7) is valid for all graphs G(V;E) with toroidal boundary graph G'(M,E'), it is possible to use Theorem 1 also in nontoroidal structures if the boundary graph is a toroidal graph.

Let $G_{2,8}$ (V,E) be a 2-dimensional toroidal grid of $N=n^2$ nodes of degree 8 obtained from $G_{2,4}$(V,E) with two diagonals in each cell (Fig. 8):

if □□□$(n^2-2n)/2$□ $f(G_{2\text{-}8},□) \geq N \min\left[\frac{2+□4□□}{2□+2+□4□□}, \frac{n-1}{□+n-1} □□□□\right]$ 13)

if □>□$(n^2-2n)/2$□ $f(G_{2,3},□) \geq \min[4+□4□□,2n]$; □ $=n^2$-□+3-

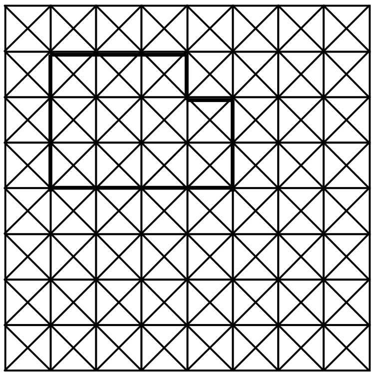


Fig 8

Proof :

It easy to verify that every partition of $G_{2\text{-}8}$(V,E) in subgrids induces a boundary graph that is a subgraph of the toroidal graph $G_{2\text{-}4}$(V,E) (Fig. 8).

In G $_{2\text{-}8}$(V,E), $n_{(□)} \geq 4+□4□□$ [7, 6]. In $G_{2,8}$(V,E) every connected component of □≥n nodes can be isolated by 2n or 2n+1 or 2n+2 edges at most (Fig 8a,b,c). The biggest component of □ nodes that can be isolated connecting 2n edges in a loop must satisfy the relation:

$4+□4□□□2n$; □□$(n-2)^2/4$ [(2)]

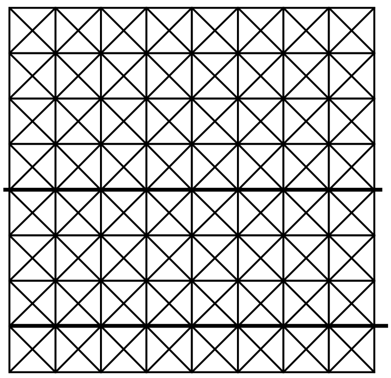


Fig 7a

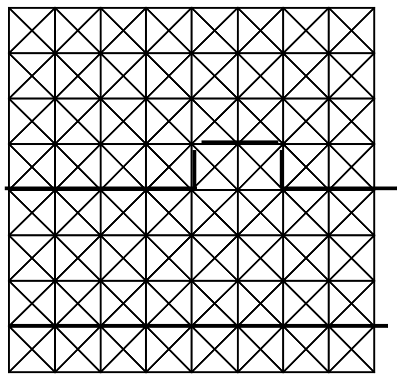


Fig 7b

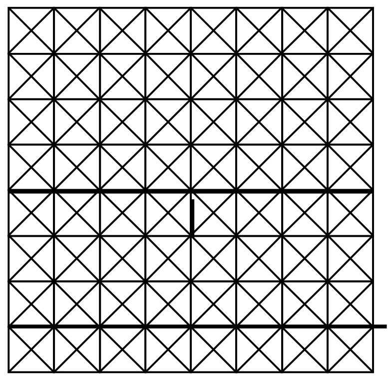


Fig 7c

as if we delete from the toroidal boundary graph a toroidal subgraph of □ nodes the remaining graph is embedded on a plan, then I($G_{2,8}$,□)= I(G $_{2,8}$,□) where □ must satisfy the relation: □+ I(G $_{2,8}$,□)= n^2-□ ;

but if 4+□4□□□2n,□>□ and I($G_{2,8}$,□)=I(G $_{2,8}$,□)≥ I(G $_{2,8}$,□) and from 6): □(□)= $\frac{2+□4□□}{2□+2+□4□□}$

---

(2) 4+□4□□> 2n if : □4□□> 2n-4 , □>(2n-4)/4, □>$(n-2)^2/4$; if n even □≥1+$(n-2)^2/4$, if n odd □≥0.75+$(n-2)^2/4$

if n even and □= □$(n-2)^2/4$□=$(n-2)^2/4$ then: 4+□4□□=4+□4 ($(n-2)^2/4$)□=4+□4((n-2)/2)□= 2n

if n odd and □= □$(n-2)^2/4$□=(($(n-2)^2/4$) –0.25 then: 4+□4□□=4+□4 ((($(n-2)^2/4$) –0.25)□□4+□4 (($(n-2)^2/4$)□=4+□4((n-2)/2)□=2n and if : 4+□4 □□> 2n and □□□(n^2-2n)/2□ I(G $_{2,6}$,□) ≥2n.

Then if □□□(n^2-2n)/2□we can apply Theorem1 with □(□)=min [(2+□4 □□)/(2□ + 2+□4 □□) , (n-1)/(□+n-1)]

If □>□(n^2-2n)/2□ it is possible to partition the grid in 2 subgrids of □ and □<□ nodes removing min[(4+□4 □□,2n] nodes at least, where □ is the minimum value such that □+4+□4 □□≥ n^2-□
□ =n^2-□+3- √(16(n^2-□+3)-64) .

The 13) can be approximated, if □□□(n^2-2n)/2□ , with:

f (G$_{2-8}$, □) ≥ N min[(1+2 □)/(□+1+2 □) , (n-1)/(□+n-1)] 14)

□

3.3 two-dimensional toroidal grid of nodes of degree 3

Let G$_{2-3}$ (V,E) be a 2-dimensional toroidal grid of N=n^2 nodes of degree 3 obtained from G$_{2,4}$(V,E) deleting n/2 edges in alternate positions of every row (Fig. 9) , then:

if N-2□- □ (6□)□≥ 0 *f (G$_{2-3}$,□) ≥ N min □ (□ 6□ □-2)/(2□ + □ 6□ □-2), □□□□ □* 15)

if N-2□- □ (6□)□< 0 *f(G$_{2,3}$,□) ≥ (□ (6□)□ where □+□+(□ (6□)□=N, □<□*

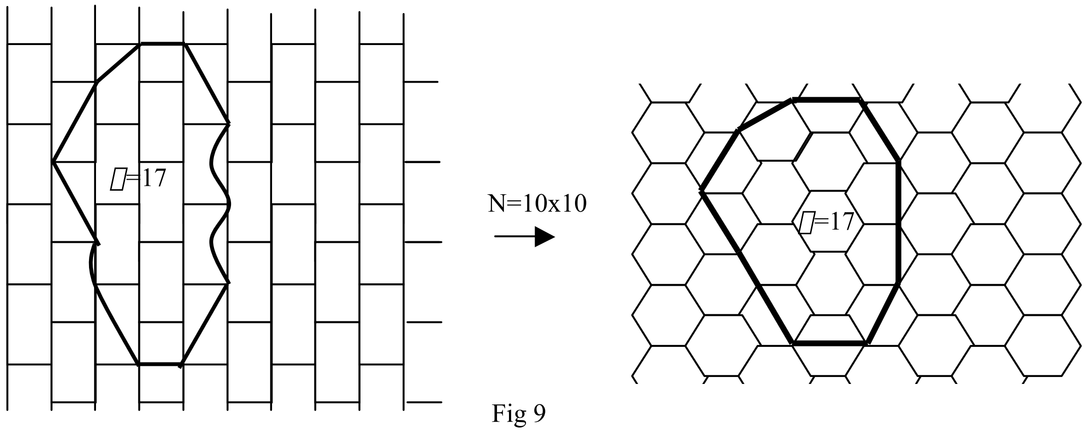


Fig 9

Proof :
in G $_{2-3}$ (V,E), I(G $_{2-3,}$□) = n(G $_{2-3,}$□) = n(□) ≥□ (6□)□ [4, 6].

In G$_{2,4}$(V,E) I(G$_{2-3,}$□) □ 2n; □ (6□)□□ 2n if 6□□ 4 n^2,
□□ 2n^2/3

but if □=2n^2/3, 2□+□ (6□)□= 2(2n^2/3)+ □ (4n^2)□=4 n^2/3+2n>n^2.
Let □$_1$ be the maximum value of □ that satisfies the relation: n^2-2□-□ (6□)□≥ 0 then:
if □□□$_1$: I(G$_{2-3,□}$)= I(G $_{2-3, □}$)=□ (6□)□ and
□(□)= (n$_{(□)}$–2)/(2□ + n$_{(□)}$ –2) = (□ 6□ □-2)/(2□+ □ 6□ □-2)

as □(□) is not a monotonically decreasing function in □:
f(G$_{2-3}$,□) ≥ N min□ (□ 6□ □-2)/(2□+ □ 6□ □-2) , □□□□□

if □>□$_1$, it is possible to partition the grid in 2 subgrids of □ and □<□ nodes removing □ (6□)□nodes, then:
f(G$_{2,3}$,□) ≥ (□ (6□)□ where □+□+(□ (6□)□=N, □>□

□

as N-2□- □ (6□)□≥ 0 if:
N-2□ ≥ □ (6□)□ 6□□N^2+4□^2 -4N□ and □□N/2
4□^2 -2□(3+2N)+ N^2≥0
□□ (3+2N- (9+12N))/4
then □$_1$ =□(3+2N- (9+12N))/4□.
The 15) can be approximated, if □□ □(3+2N- (9+12N))/4□with : f(G$_{2-3}$,□) ≥ N (6□ -2)/(2□+ 6□ -2) . 16)

4. two-dimension non-toroidal structures

Let $\underline{G}_{2\text{-}4}$(V,E) be a 2-dimensional grid defined as Cartesian product of 2 paths of n vertices.

If in $\underline{G}_{2\text{-}4}$(V,E), □V□=$n^2$ it is necessary to delete at least f ($\underline{G}_{2\text{-}4}$, □)$_{min}$ nodes to obtain a graph with a maximal connected component of □ nodes or less, then also in the toroidal graph $G_{2\text{-}4}$(V',E), □V'□=$(n+1)^2$ it is always possible to obtain the same result removing f($\underline{G}_{2\text{-}4}$, □)+2n nodes; see the example of Fig. 10 :

$$f(\underline{G}_{2\text{-}4}(V,E), □) + 2n \geq f(G_{2\text{-}4}(V',E), □)$$

then :

$$f(\underline{G}_{2\text{-}4}(V,E), □) \geq f(G_{2\text{-}4}(V',E), □) - 2n$$

as f ($G_{2\text{-}4}$(V',E), □) =f ($G_2(n+1)^2$,E), □) $\geq (n+1)^2$ min□ (□2 (2□-1)□)/(2□ +(□2 (2□-1)□)) , □□□□□:

f ($\underline{G}_{2\text{-}4}$,□) $\geq (n+1)^2$ min□ (□2 (2□-1)□)/(2□ +(□2 (2□-1)□)) , □□□□□ -2n-1

that can be approximated by:

if □<□(n-1) ^{2}/2□: f ($\underline{G}_{2\text{-}4}$,□) $\geq (n+1)^2$ □ (2□-1) / [□+ (2□-1)]□ -2n

The Khanna-Fuchs value is : (N- (aN))/(1+ a).

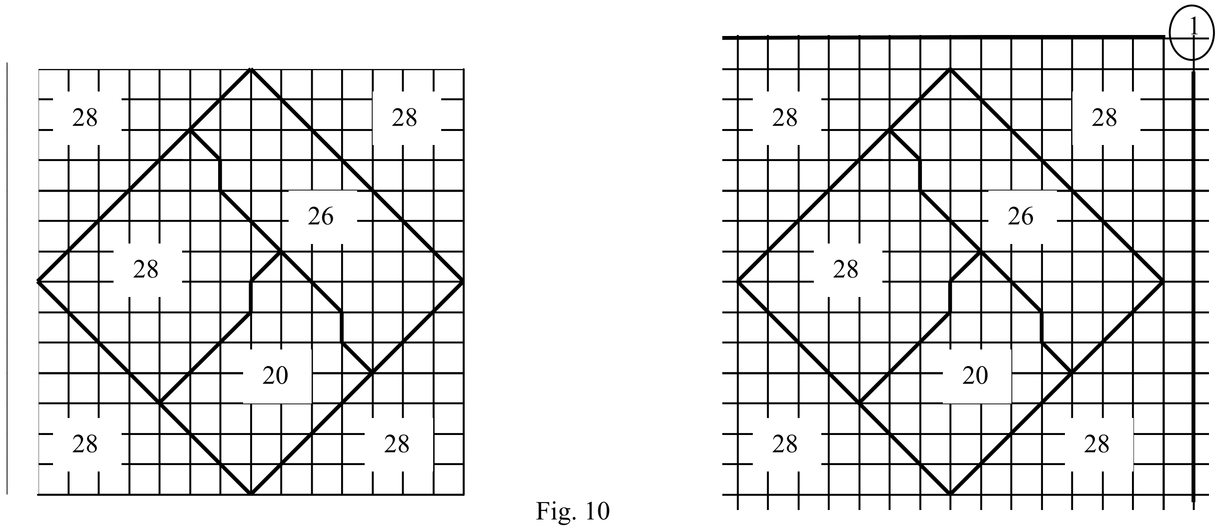


Fig. 10

Analogously for the other structures: (Fig. 11, 12, 13):

3+□ (12□-3)□<2n, □□□n^2/3-4n/3+19/12□-1: f($\underline{G}_{2\text{-}6}$,□) ≥ $(n+1)^2$ □[1+ (12□-3)]/[2□+1+ (12□-3)]□ -2n-1

4+□4 □□□ 2n , □□(n-2) ^{2}/4: f($\underline{G}_{2\text{-}8}$,□) ≥ $(n+1)^2$ □(1+2 □)/(□+1+2 □)□-2n-1

N-2□ ≥ □ (6□)□ □□ (3+2N- (9+12N))/4: f ($\underline{G}_{2\text{-}3}$,□) ≥ $(n+1)^2$ □[(6□)–2]/[2□ + (6□)–2]□-3n/2

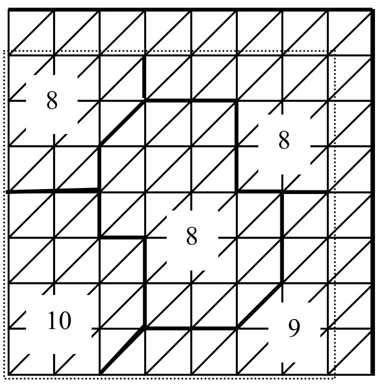


Fig 11

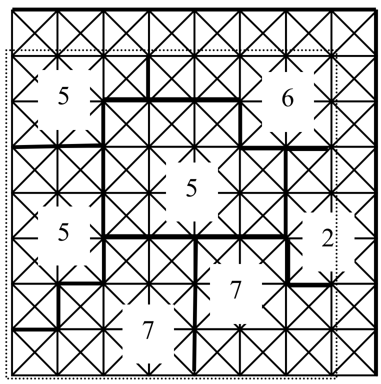


Fig 12

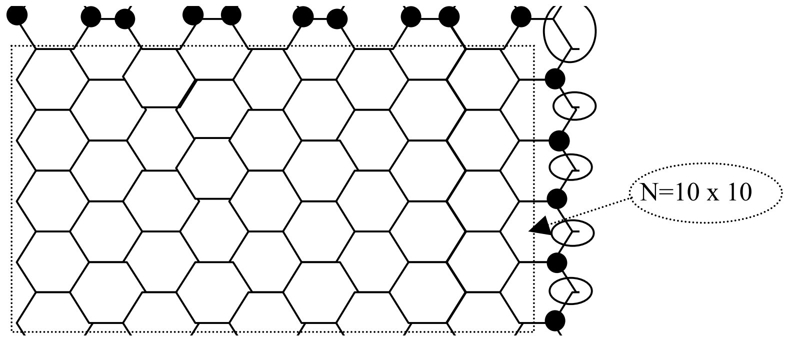


Fig 13

5. Conclusions

Using the Euler characteristic equation for polyhedra of genus □□1 and an edges isoperimetric relation (the lower bound on the edges that must be used to connect the nodes that must be removed to isolate a subgraph of □ nodes) we have obtained the functions that approximate, and in some cases represent, the minimal number of nodes that must be removed in a toroidal regular graph to decompose the graph in connected components of □ nodes or less.
As every graph is embeddible on some orientable surface and any embedding on an orientable surface with a minimum number of handles has all its faces simply connected, a future work may be to apply the Euler equation for polyhedra of genus □>1 to boundary graphs that can be obtained from non toroidal structures like a n-dimensional cube or n-dimension grids.

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