On the derivative of the stress-strain relation in a no-tension material

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Abstract

The stress–strain relation of a no–tension material, used to model masonry structures, is determined by the nonlinear projection of the strain tensor onto the image of the convex cone of negative–semidefinite stresses under the fourth–order tensor of elastic compliances. We prove that the stress–strain relation is indefinitely differentiable on an open dense subset $\mathcal{O}$ of the set of all strains. The set $\mathcal{O}$ consists of four open connected regions determined by the rank $k = 0, 1, 2, 3$ of the resulting stress. Further, an equation for the derivative of the stress–strain relation is derived. This equation cannot be solved explicitly in the case of a material of general symmetry, but it is shown that for an isotropic material this leads to the derivative established earlier in [14], [16] by different means. For a material of general symmetry, when the tensor of elasticities does not have the representation known in the isotropic case, only general steps leading to the evaluation of the derivative are described.

1 Introduction

This note deals with theoretical aspects of a model treating the masonry structure as a nonlinear elastic material with zero tensile strength and infinite compressive strength [11], [12], [5], [9], [2], [4], [1]. The resulting constitutive equation, known as the equation of *masonry–like* or *no–tension* materials, accounts for some of masonry's peculiarities, in particular, its incapability to withstand tensile stresses. Its nonlinear stress–strain relation is determined by the nonlinear projection $\mathbb{P}$ of the strain tensor onto the image of the set $\mathbb{L}^{-1}\,\mathrm{Sym}^-$ of negative–semidefinite stresses Sym^- under the fourth–order tensor of elastic compliances $\mathbb{L}^{-1}$ with respect to the energetic scalar product on the space of symmetric tensors.[1]

No–tension materials provide a particular case of saturated elastic materials introduced later by Marcelo Epstein [7]; the stored energy of no–tension materials is the relaxed energy of saturated materials, see Epstein [6], [7] and Epstein & Forcinito [8]. Positive–semidefinite effective stresses in wrinkling membranes [6], [7] and their constitutive equations differ just by the sign from stresses and constitutive equations of masonry materials.

Practical applications of no–tension materials involve numerical implementation. The constitutive model is combined with the finite–element method in the code NOSA–ITACA [20] to provide a tool for studying the structural behavior of existing masonry structures in both the static and dynamical situations [13], [15], [3]. The code has been successfully applied to the analysis of some buildings of historical interest [16]. A substantial ingredient of the numerical solution of the equilibrium problem is the derivative of the stress–strain relation, allowing to determine the tangent stiffness matrix required by the Newton-Raphson method for solving the nonlinear algebraic system resulting from the discretization into finite elements. In an isotropic material the derivative, as well as the solution to the constitutive equation, can be determined explicitly [16, Section 2.4].

In this note we deal with materials of arbitrary symmetry and apply our general results to isotropic materials to show the coincidence with results derived previously for isotropic materials. We prove that the stress–strain relation is indefinitely differentiable on an open dense subset $\mathcal{O}$ of the set of all strains and derive an equation (see (12), below) for the derivative of the stress with respect to strain. The set $\mathcal{O}$ consists of four open connected regions determined by the rank $k = 0,\ 1,\ 2,\ 3$ of the resulting stress, with the cases $k = 0$ and $k = 3$ being trivial. The mentioned equation (12) determines the derivative only implicitly, for two reasons: first, it involves an orthogonal projection onto the tangent space to the set of all elastic strains relative to the energetic scalar product (determined by the tensor of elastic constants) which cannot be determined explicitly and, second, it involves an anticommutation relation which cannot be solved by a closed form formula except for the case of isotropic materials. For a material of general symmetry, when the tensor of elasticities does not have the representation known in the isotropic case, only general steps leading to the evaluation of the derivative can be described.

To describe the idea of the proof and the line of our arguments, we note that we employ the characterization of the stress–strain relation by the above mentioned nonlinear projection $\mathbb{P}$ of the strain tensor onto the image $\mathbb{L}^{-1}\,\mathrm{Sym}^-$ of the convex cone of negative–semidefinite stresses under the fourth–order tensor of elastic compliances. The proofs of our general results are based on those [19]

[1]We refer to Section 2 for the notation and further details.

on the differentiability and the derivative of the nonlinear orthogonal projection onto a closed convex set whose boundary contains a hierarchy of manifolds of singular points of various orders (such as corners, edges, faces). Indeed, the set $\mathbb{L}^{-1}\,\mathrm{Sym}^-$ is a closed convex cone with nonempty interior. Its boundary in the six–dimensional space of symmetric tensors is *piecewise* smooth in the sense that it consists of "a corner," which is the zero tensor, and further of "edges," and "faces."[2] These sets are the images $\mathbb{L}^{-1}\,\mathrm{Sym}_0^- \equiv \{0\}$, $\mathbb{L}^{-1}\,\mathrm{Sym}_1^-$, $\mathbb{L}^{-1}\,\mathrm{Sym}_2^-$, under $\mathbb{L}^{-1}$, of the sets of negative–semidefinite tensors $\mathrm{Sym}_0^- \equiv \{0\}$, Sym_1^-, Sym_2^-, of ranks 0, 1, and 2, respectively. Each of the last three sets is an indefinitely differentiable manifold. By the first main result of [19], this implies that the projection $\mathbb{P}$ (and hence also the stress) is indefinitely differentiable on the interiors W_0, W_1, and W_2 of the set of all strains V_0, V_1, and V_2 that are mapped by $\mathbb{P}$ into the sets $\mathbb{L}^{-1}\,\mathrm{Sym}_0^-$, $\mathbb{L}^{-1}\,\mathrm{Sym}_1^-$, $\mathbb{L}^{-1}\,\mathrm{Sym}_2^-$, respectively. By the second main result of [19], the derivative of $\mathbb{P}$ on each of the sets W_0, W_1, and W_2 is related to the second fundamental form (i.e., the curvature) of the manifolds $\mathbb{L}^{-1}\,\mathrm{Sym}_0^-$, $\mathbb{L}^{-1}\,\mathrm{Sym}_1^-$, $\mathbb{L}^{-1}\,\mathrm{Sym}_2^-$. The main steps in the proof is thus the determination of the nature of the sets $\mathbb{L}^{-1}\,\mathrm{Sym}_0^-$, $\mathbb{L}^{-1}\,\mathrm{Sym}_1^-$, $\mathbb{L}^{-1}\,\mathrm{Sym}_2^-$, the evaluation of the second fundamental form of these sets (in Section 3) and the maps associated with it (in Section 4). The main differentiability results are stated in Section 5 for materials of general symmetry and in Section 6 for isotropic materials.

2 No–tension materials

Throughout, Lin denotes the set of all second order tensors on $\mathbb{R}^n$, i.e., linear transformations from $\mathbb{R}^n$ into itself where n is an arbitrary positive integer; typically $n = 2$ (planar no–tension bodies) or $n = 3$ (full fledged no–tension bodies). Sym is the subspace of symmetric tensors, Sym^+ the set of all positive–semidefinite elements of Sym, Sym^- is the set of all negative–semidefinite elements of Sym. The scalar product of $A, B \in \mathrm{Lin}$ is defined by $A \cdot B = \mathrm{tr}(AB^{\mathrm{T}})$ and $|\cdot|$ denotes the associated euclidean norm on Lin. We denote by $\mathrm{I} \in \mathrm{Lin}$ the unit tensor.

We interpret the fourth–order tensors as linear transformations from Sym into itself. We denote by $\mathbb{I}$ the fourth–order identity tensor, given by $\mathbb{I}A = A$ for every $A \in \mathrm{Sym}$. Given the symmetric tensors A and B, we denote by $A \otimes B$ the fourth–order tensor defined by $A \otimes B[H] = (B \cdot H)A$ for $H \in \mathrm{Sym}$.

We denote the maps from Sym into Sym, linear or not, by outlined letters $\mathbb{L}$, $\mathbb{P}$, etc. We often inclose the arguments of linear transformations from Sym to Sym (i.e. of fourth–order tensors) in square brackets.

To describe the stress, we assume that $\mathbb{L} : \mathrm{Sym} \to \mathrm{Sym}$ is a given fourth

[2] In a way similar to the corner, edges, and faces of the boundary of the octant of vectors with nonnegative components in the three–dimensional space.

–order tensor of elastic constants, such that

$$\left.\begin{array}{r} A\cdot \mathbb{L}A>0 \quad \text{for all } A\in \mathrm{Sym}, A\neq 0,\\ B\cdot \mathbb{L}C=C\cdot \mathbb{L}B \quad \text{for all } B,C\in \mathrm{Sym}\,. \end{array}\right\} \tag{1}$$

Throughout the paper we assume that $\mathbb{L}$ is a fixed linear transformation satisfying (1).

Definition 2.1. We define the energetic scalar product on Sym by setting $(A,B)=A\cdot \mathbb{L}B$ for any A, $B\in \mathrm{Sym}$; we further denote by $\|A\|:=\sqrt{(A,A)}$ the energetic norm.

Proposition 2.2. *If* $X\in \mathrm{Sym}$, *there exists a unique triplet* (T,Y,Z) *of elements of* Sym *satisfying the following three equivalent statements:*
(i) *we have*

$$\left.\begin{array}{c} X=Y+Z,\\ T=\mathbb{L}Y,\\ T\in \mathrm{Sym}^-,\quad Z\in \mathrm{Sym}^+,\\ T\cdot Z=0. \end{array}\right\} \tag{2}$$

(ii) *we have* $(2)_{1,2}$ *and*

$$\left.\begin{array}{c} T\in \mathrm{Sym}^-,\\ (T-T^*)\cdot Z\geq 0 \quad \textit{for each } T^*\in \mathrm{Sym}^-\,. \end{array}\right\}$$

(iii) *we have* $(2)_{1,2}$ *and* Y *is the metric projection of* X *onto the convex cone* $\mathbb{L}^{-1}\,\mathrm{Sym}^-$ *with respect to the energetic scalar product, i.e.,* $Y\in \mathbb{L}^{-1}\,\mathrm{Sym}^-$ *satisfies*

$$\|Y-X\|=\min\{\|B-X\|: B\in \mathbb{L}^{-1}\,\mathrm{Sym}^-\}.$$

We refer to [2], [9] and [4] for various forms of the above statement and the proof.

If (T,Y,Z) is the triplet associated with X in this proposition, we define by $\mathbb{P}:\mathrm{Sym}\to\mathrm{Sym}$ the metric projection onto $\mathbb{L}^{-1}\,\mathrm{Sym}^-$ mentioned in (iii); we call $Y=\mathbb{P}(X)$ the elastic part of the deformation corresponding to the total deformation $X\in\mathrm{Sym}$ and $Z=X-\mathbb{P}(X)$ the fracture part of the deformation. The stress $\mathbb{T}:\mathrm{Sym}\to\mathrm{Sym}$ and stored energy $\hat{W}:\mathrm{Sym}\to\mathbb{R}$ are given by

$$\mathbb{T}(X)=T=\mathbb{L}\mathbb{P}(X),\quad \hat{W}(X)=\tfrac{1}{2}\mathbb{T}(X)\cdot X$$

for any $X\in\mathrm{Sym}$. When $\mathbb{L}$ is isotropic, the explicit form of the response function $\mathbb{T}$ and its further analysis, first given in [13], [14], is presented in Section 6, below. Generally, the map $\mathbb{T}$ is monotone and Lipschitz continuous and the function $\hat{W}$ is continuously differentiable, convex and $\mathrm{D}\,\hat{W}=\mathbb{T}$; see [4, Proposition 4.4 and Lemma 5.1].

3 The second fundamental form of $\mathbb{L}^{-1}\,\mathrm{Sym}_k^-$

Throughout, let k be an integer with $0 \le k \le n$. For each $X \in \mathrm{Sym}$ we denote by $Q(X)$ and $R(X)$ the projectors onto $\operatorname{ran} X := \{Xx \in \mathbb{R}^n : x \in \mathbb{R}^n\}$ and $\ker X := \{x \in \mathbb{R}^n : Xx = 0\}$, respectively, $Q(X) + R(X) = \mathrm{I}$. We denote by Sym_k and Sym_k^- the set of all elements X of Sym and Sym^-, respectively, with $\operatorname{rank} X = k$. We denote by Q_k and R_k the restrictions of Q and R to Sym_k^-.

For each $X \in \mathrm{Sym}$ there exists a unique $X^{-1} \in \mathrm{Sym}$ such that

$$X^{-1}X = X\,X^{-1} = Q(X). \tag{3}$$

Indeed, since X maps $\operatorname{ran} X$ bijectively onto $\operatorname{ran} X$, we can put

$$X^{-1} = [X|\operatorname{ran} X]^{-1} Q(X)$$

where $[X|\operatorname{ran} X]^{-1}$ is the standard inverse of an injective map. Then clearly (3) hold. This proves the existence. The uniqueness is clear. Note that given the spectral representation of X with the eigenvalues x_i then X^{-1} has the same spectral representation with eigenvalues

$$\xi_i = \begin{cases} 1/x_i & \text{if} \quad x_i \ne 0, \\ 0 & \text{if} \quad x_i = 0. \end{cases}$$

Definitions 3.1.

(i) We define the tangent space $\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y)$ to $\mathbb{L}^{-1}\,\mathrm{Sym}_k^-$ at $Y \in \mathbb{L}^{-1}\,\mathrm{Sym}_k^-$ as the set of all $B \in \mathrm{Sym}$ such that there exists a continuously differentiable map A satisfying

$$A : (-\epsilon, \epsilon) \to \mathbb{L}^{-1}\,\mathrm{Sym}_k^-, \quad A(0) = Y, \quad \dot{A}(0) = B. \tag{4}$$

(ii) We define the normal space $\mathrm{Nor}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y)$ to $\mathbb{L}^{-1}\,\mathrm{Sym}_k^-$ at $Y \in \mathbb{L}^{-1}\,\mathrm{Sym}_k^-$ as the orthogonal complement of $\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y)$ in Sym relative to the energetic scalar product.

(iii) If $S : \mathbb{L}^{-1}\,\mathrm{Sym}_k^- \to V$ is a map on $\mathbb{L}^{-1}\,\mathrm{Sym}_k^-$ with values in a finite dimensional vectorspace, we say that S is differentiable at $Y \in \mathbb{L}^{-1}\,\mathrm{Sym}_k^-$ if there exists a linear map $\mathrm{D}\,S(Y) : \mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y) \to V$ such that

$$\mathrm{D}\,S(Y)[B] = \frac{d}{dt} S(A(t))\big|_{t=0}$$

for any continuously differentiable curve A as in (4). We do not indicate graphically the fact that $\mathrm{D}\,S(Y)[\cdot]$ is the surface derivative relative to $\mathbb{L}^{-1}\,\mathrm{Sym}_k^-$ as this is uniquely given by the domain of S.

Proposition 3.2.

(i) *The set* $\mathbb{L}^{-1}\,\mathrm{Sym}_k^-$ *is a connected indefinitely differentiable manifold of dimension* $\frac{1}{2}k(2n - k + 1)$;

(ii) *if* $T \in \mathrm{Sym}_k^-$ *then*

$$\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, \mathbb{L}^{-1}T) = \{\mathbb{L}^{-1}B \in \mathrm{Sym} : R_k(T)BR_k(T) = 0\}, \tag{5}$$

$$\mathrm{Nor}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, \mathbb{L}^{-1}T) = \{Z \in \mathrm{Sym} : R_k(T)ZR_k(T) = Z\}. \tag{6}$$

Proof (i): By [10, Proposition 1.1, Section 5.1] Sym_k^- is a connected manifold of the indicated dimension and $\mathbb{L}^{-1}\,\mathrm{Sym}_k^-$ is its image under a bijective transformation. (ii): It follows from the results of [19] that

$$\mathrm{Tan}(\mathrm{Sym}_k^-, T) = \{B \in \mathrm{Sym} : R_k(T)BR_k(T) = 0\}.$$

Equations (5) and (6) then follow. □

Lemma 3.3. *The map Q_k is indefinitely differentiable on Sym_k^- and its surface derivative is given by*

$$\mathrm{D}\,Q_k(T)[B]Q_k(T) = R_k(T)BT^{-1} \tag{7}$$

for any $T \in \mathrm{Sym}_k^-$ and any $B \in \mathrm{Tan}(\mathrm{Sym}_k^-, T)$.

Proof See [19]. □

Definitions 3.4.

(i) For any $Y \in \mathbb{L}^{-1}\,\mathrm{Sym}_k^-$, we denote the projections onto $\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y)$ and $\mathrm{Nor}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y)$ by $\mathbb{Q}_k(Y)$ and $\mathbb{R}_k(Y)$, respectively, and observe that

$$(\mathbb{Q}_k(\mathbb{L}^{-1}T) - \mathbb{I})\mathbb{L}^{-1}\big(C - R_k(T)CR_k(T)\big) = 0 \tag{8}$$

for any $T \in \mathrm{Sym}_k^-$ and $C \in \mathrm{Sym}$.

(ii) We define the second fundamental form $\mathbb{B}_k$ of $\mathbb{L}^{-1}\,\mathrm{Sym}_k^-$ as a map which associates with each $Y \in \mathbb{L}^{-1}\,\mathrm{Sym}_k^-$ a bilinear form $\mathbb{B}_k(Y) : \mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y) \times \mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y) \to \mathrm{Nor}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y)$ given by

$$\mathbb{B}_k(Y)(B, C) = \mathrm{D}\,\mathbb{Q}_k(Y)[B]C$$

for every $B,\ C \in \mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y)$.

Lemma 3.5. *If $T \in \mathrm{Sym}_k^-$ and $B,\ C \in \mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, \mathbb{L}^{-1}T)$ then*

$$\begin{aligned} &\mathrm{D}\,\mathbb{Q}_k(\mathbb{L}^{-1}T)[B]C \\ &\quad = \mathbb{R}_k(\mathbb{L}^{-1}T)\Big[\mathbb{L}^{-1}\big[R_k(T)(\mathbb{L}[B]T^{-1}\mathbb{L}[C] + \mathbb{L}[C]T^{-1}\mathbb{L}[B])R_k(T)\big]\Big] \end{aligned} \tag{9}$$

which also gives the second fundamental form $\mathbb{B}_k(\mathbb{L}^{-1}T)(B, C)$ of $\mathbb{L}^{-1}\,\mathrm{Sym}_k^-$.

Proof Differentiating (8) in the direction $\mathbb{L}^{-1}B \in \mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, \mathbb{L}^{-1}T)$ and using $\mathbb{R}_k(\mathbb{L}^{-1}T) = \mathbb{I} - \mathbb{Q}_k(\mathbb{L}^{-1}T)$ we obtain for any $C \in \mathrm{Sym}$ the relation

$$\begin{aligned} &\mathrm{D}\,\mathbb{Q}_k(\mathbb{L}^{-1}T)[\mathbb{L}^{-1}B]\mathbb{L}^{-1}\big(C - R_k(T)CR_k(T)\big) \\ &\qquad = \mathbb{R}_k(\mathbb{L}^{-1}T)\mathbb{L}^{-1}\big[\mathrm{D}\,R_k(T)[B]CR_k(T) + R_k(T)C\,\mathrm{D}\,R_k(T)[B]\big] \end{aligned}$$

For C satisfying $R_k(T)CR_k(T) = 0$, i.e., $Q_r(T)C = C$ this reduces to

$$\begin{aligned}&\mathrm{D}\mathbb{Q}_k(\mathbb{L}^{-1}T)[\mathbb{L}^{-1}B]\mathbb{L}^{-1}C\\ &\quad = \mathbb{R}_k(\mathbb{L}^{-1}T)\mathbb{L}^{-1}\big[\mathrm{D}\,R_k(T)[B]Q_k(T)CR_k(T) + R_k(T)CQ_k(T)\,\mathrm{D}\,R_k(T)[B]\big].\end{aligned}$$

Combining with (7) we obtain

$$\mathrm{D}\,\mathbb{Q}_k(\mathbb{L}^{-1}T)[\mathbb{L}^{-1}B]\mathbb{L}^{-1}C = \mathbb{R}_k(\mathbb{L}^{-1}T)\mathbb{L}^{-1}\big[R_k(T)(BT^{-1}C + CT^{-1}B)R_k(T)\big].$$

Replacing B by $\mathbb{L}[B]$ and C by $\mathbb{L}[C]$ where now B, $C \in \mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y)$, we obtain (9). □

4 The normal cone to $\mathbb{L}^{-1}\,\mathrm{Sym}^-$

If $Y \in \mathbb{L}^{-1}\,\mathrm{Sym}^-$, we define the normal cone $\mathrm{Nor}^+(\mathbb{L}^{-1}\,\mathrm{Sym}^-, Y)$ by

$$\mathrm{Nor}^+(\mathbb{L}^{-1}\,\mathrm{Sym}^-, Y) = \{Z \in \mathrm{Sym} : (Z, V - Y) \le 0 \text{ for all } V \in \mathbb{L}^{-1}\,\mathrm{Sym}^-\}$$

where we use the energetic scalar product.

Proposition 4.1. *If $T \in \mathrm{Sym}_k^-$ then*

$$\mathrm{Nor}^+(\mathbb{L}^{-1}\,\mathrm{Sym}^-, \mathbb{L}^{-1}T) = \{Z \in \mathrm{Sym}^+ : R_k(T)ZR_k(T) = Z\}.$$

Proof Since $\mathbb{L}^{-1}\,\mathrm{Sym}^-$ is a convex cone, $\mathrm{Nor}^+(\mathbb{L}^{-1}\,\mathrm{Sym}^-, Y)$ is the set of all elements of the dual cone that are perpendicular to Y (see [17, Example 11.4(b)]). The dual cone with respect to the energetic scalar product to $\mathbb{L}^{-1}\,\mathrm{Sym}^-$ is Sym^+ and thus

$$\mathrm{Nor}^+(\mathbb{L}^{-1}\,\mathrm{Sym}^-, Y) = \{Z \in \mathrm{Sym}^+ : (Z, Y) = 0\};$$

however, since $Z \in \mathrm{Sym}^+$ and $Y \in \mathbb{L}^{-1}\,\mathrm{Sym}^-$, the relation $(Z, Y) = 0$ implies $ZT = 0$; this in turn implies that $ZQ_k(T) = 0$. We finally conclude that $R_k(T)ZR_k(T) = Z$. □

For any $Y \in \mathbb{L}^{-1}\,\mathrm{Sym}_k^-$ and $Z \in \mathrm{Nor}^+(\mathbb{L}^{-1}\,\mathrm{Sym}^-, Y)$, denote by $\mathbb{C}_k(Y, Z)$ the linear transformation from $\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y)$ into itself such that

$$(\mathbb{C}_k(Y, Z)B, C) = (Z, \mathbb{B}_k(Y)(B, C))$$

for all B, $C \in \mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y)$.

Proposition 4.2. *For each $T \in \mathrm{Sym}_k$, each $Z \in \mathrm{Nor}^+(\mathbb{L}^{-1}\,\mathrm{Sym}^-, Y)$ and each $B \in \mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y)$, with $Y = \mathbb{L}^{-1}T$, we have*

$$\mathbb{C}_k(\mathbb{L}^{-1}T, Z)B = \mathbb{Q}_k(\mathbb{L}^{-1}T)\big[T^{-1}\mathbb{L}[B]Z + Z\mathbb{L}[B]T^{-1}\big].$$

Proof Letting $C \in \operatorname{Tan}(\mathbb{L}^{-1} \operatorname{Sym}_k^-, Y)$, we find from Lemma 3.5 that

$$\begin{aligned}
&(Z, \mathrm{D}\mathbb{Q}_k(\mathbb{L}^{-1}T)[B]C) \\
&\quad = (Z, \mathbb{R}_k(\mathbb{L}^{-1}T)\mathbb{L}^{-1}\big[R_k(T)(\mathbb{L}[B]T^{-1}\mathbb{L}C + \mathbb{L}[C]T^{-1}\mathbb{L}[B])R_k(T)\big]) \\
&\quad = (Z, \mathbb{L}^{-1}\big[R_k(T)(\mathbb{L}[B]T^{-1}\mathbb{L}[C] + \mathbb{L}[C]T^{-1}\mathbb{L}[B])R_k(T)\big]) \\
&\quad = Z \cdot \big[R_k(T)(\mathbb{L}[B]T^{-1}\mathbb{L}[C] + \mathbb{L}[C]T^{-1}\mathbb{L}[B])R_k(T)\big] \\
&\quad = Z \cdot (\mathbb{L}[B]T^{-1}\mathbb{L}[C] + \mathbb{L}[C]T^{-1}\mathbb{L}[B]) \\
&\quad = Z \cdot \mathbb{L}[B]T^{-1}\mathbb{L}[C] + Z \cdot \mathbb{L}[C]T^{-1}\mathbb{L}[B] \\
&\quad = T^{-1}\mathbb{L}[B]Z \cdot \mathbb{L}[C] + Z\mathbb{L}[B]T^{-1} \cdot \mathbb{L}[C]. \quad \square
\end{aligned}$$

5 The main results: the differentiability and the derivative of the stress

We say that a map $\mathbb{F} : \operatorname{Sym} \to \operatorname{Sym}$ is differentiable at $X \in \operatorname{Sym}$ if there exists a linear transformation $\mathbb{D}$ from Sym into itself such that

$$\lim_{B \to X} \|\mathbb{F}(B) - \mathbb{F}(X) - \mathbb{D}(B - X)\| / \|B - X\| = 0.$$

We call $\mathbb{D}$ the derivative of $\mathbb{F}$ at X and write $\mathrm{D}\,\mathbb{F}(X)[H] = \mathbb{D}H$ for each $H \in \operatorname{Sym}$.

Define the sets

$$V_k := \{X \in \operatorname{Sym} : \mathbb{L}\mathbb{P}(X) \in \operatorname{Sym}_k^-\}, \tag{10}$$

$0 \le k \le n$; it is easy to see that

$$V_k = \bigcup \{Y + \operatorname{Nor}^+(\mathbb{L}^{-1}\operatorname{Sym}^-, Y) : Y \in \mathbb{L}^{-1}\operatorname{Sym}_k^-\}$$

and clearly,

$$\bigcup_{k=0}^{n} V_k = \operatorname{Sym}.$$

Furthermore, define W_k as the interior of V_k and put

$$\mathcal{O} = \bigcup_{k=0}^{n} W_k.$$

It is easy to see that $\mathcal{O}$ is an open dense subset of Sym. Recalling that the set $\mathbb{L}^{-1}\operatorname{Sym}_k^-$ is an indefinitely differentiable manifold (Proposition 3.2) and using [19, Theorem 1.6] we obtain the following result:

Theorem 5.1. *The map* $\mathbb{P}$ (*and hence also* $\mathbb{T}$) *is indefinitely differentiable on* $\mathcal{O}$.

Furthermore, [19, Theorem 2.3.4] gives the following.

Theorem 5.2. *For every* $X \in W_k$ *and* $C \in \mathrm{Sym}$ *we have*

$$\mathrm{D}\,\mathbb{P}(X)[C] = \left[\mathbb{I}_k(Y) - \mathbb{C}_k(Y, X - Y)\right]^{-1} \mathbb{Q}_k(Y)C \tag{11}$$

where $Y = \mathbb{P}(X)$ *and* $\mathbb{I}_k(Y)$ *is the identity transformation on* $\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y)$.

We note that the existence of the inverse follows from the negative–semidefinite character of $\mathbb{C}_k(Y, X - Y)$, which in turn is a consequence of the convexity of $\mathbb{L}^{-1}\,\mathrm{Sym}^-$, see [19, Theorem 2.3.4(i)].

A combination of (11) with Proposition 4.2 leads to the following relation for the derivative:

Theorem 5.3. *If* $X \in W_k$ *and* $C \in \mathrm{Sym}$ *then* $\mathrm{D}\,\mathbb{P}(X)[C] = B$ *where* $B \in \mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, \mathbb{P}(X))$ *is the unique solution of the equation*

$$B - \mathbb{Q}_k(\mathbb{L}^{-1}T)\big[T^{-1}\mathbb{L}[B]Z + Z\mathbb{L}[B]T^{-1}\big] = \mathbb{Q}_k(\mathbb{L}^{-1}T)C \tag{12}$$

where $T = \mathbb{L}\mathbb{P}(X)$, $Z = X - \mathbb{P}(X)$.

6 The isotropic case

Let us consider the isotropic elasticity tensor

$$\mathbb{L} = 2\mu\mathbb{I} + \lambda \mathrm{I} \otimes \mathrm{I},$$

with μ and λ the Lamé moduli of the material, satisfying the conditions $\mu > 0$, $2\mu + 3\lambda > 0$ which guarantee that Conditions (1) are satisfied. In particular, $\mathbb{L}$ is invertible and

$$\mathbb{L}^{-1} = \frac{1}{2\mu}\mathbb{I} - \frac{\lambda}{2\mu(2\mu + 3\lambda)}\mathrm{I} \otimes \mathrm{I}.$$

For $X \in \mathrm{Sym}$, let $x_1 \le x_2 \le x_3$ be its ordered eigenvalues and q_1, q_2, q_3 the corresponding eigenvectors. We introduce the orthonormal basis of Sym (with respect to the scalar product "$\cdot$")

$$O_{11} = q_1 \otimes q_1,\ O_{22} = q_2 \otimes q_2,\ O_{33} = q_3 \otimes q_4, \tag{13}$$

$$O_{12} = \frac{1}{\sqrt{2}}(q_1 \otimes q_2 + q_2 \otimes q_1),\ O_{13} = \frac{1}{\sqrt{2}}(q_1 \otimes q_3 + q_3 \otimes q_1),$$

$$O_{23} = \frac{1}{\sqrt{2}}(q_2 \otimes q_3 + q_3 \otimes q_2),$$

where, for a and b vectors, the diade $a \otimes b$ is defined by $a \otimes bh = (b \cdot h)a$, for any vector h and $\cdot$ is the scalar product in the space of vectors.

Given X, the projection $Y = \mathbb{P}(X)$ onto the convex cone $\mathbb{L}^{-1}\mathrm{Sym}^-$ with respect to the energetic scalar product can be calculated explicitly [16]. In particular,

$$\left.\begin{array}{llll}\text{if } X \in V_0 & \text{then} & \mathbb{P}(X) = 0, \\ \text{if } X \in V_1 & \text{then} & \mathbb{P}(X) = x_1 O_{11} - \dfrac{\alpha}{2(1+\alpha)} x_1 (O_{22} + O_{33}), \\ \text{if } X \in V_2 & \text{then} & \mathbb{P}(X) = x_1 O_{11} + x_2 O_{22} - \dfrac{\alpha}{2+\alpha}(x_1 + x_2) O_{33}, \\ \text{if } X \in V_3 & \text{then} & \mathbb{P}(X) = X, \end{array}\right\} \qquad (14)$$

where $\alpha = \lambda/\mu$ and the sets V_k introduced in (10) are

$$\left.\begin{array}{l} V_0 = \{X \in \mathrm{Sym} \,:\, x_1 \geq 0\}\,, \\ V_1 = \{X \in \mathrm{Sym} \,:\, x_1 < 0,\ \alpha x_1 + 2(1+\alpha) x_2 \geq 0\}\,, \\ V_2 = \{X \in \mathrm{Sym} \,:\, \alpha x_1 + 2(1+\alpha) x_2 < 0,\ 2x_3 + \alpha \,\mathrm{tr}\, X \geq 0\}\,, \\ V_3 = \{X \in \mathrm{Sym} \,:\, 2x_3 + \alpha \,\mathrm{tr}\, X < 0\}\,. \end{array}\right\} \qquad (15)$$

Thus, setting $T = \mathbb{L}[\mathbb{P}(X)]$, from $(14)_1$–$(14)_4$ we get the explicit expression of the stress tensor T, with $E = \mu(2\mu + 3\lambda)/(\mu + \lambda)$ the Young modulus:

$$\left.\begin{array}{lll} \text{if } X \in V_0 & \text{then} & T = 0 \in \mathrm{Sym}_0^-\,, \\ \text{if } X \in V_1 & \text{then} & T = E x_1 O_{11} \in \mathrm{Sym}_1^-\,, \\ \text{if } X \in V_2 & \text{then} & T = \dfrac{2\mu}{2+\alpha}\,\{[2(1+\alpha)x_1 + \alpha x_2] O_{11} \\ & & \quad +[2(1+\alpha)x_2 + \alpha x_1] O_{22}\} \in \mathrm{Sym}_2^-\,, \\ \text{if } X \in V_3 & \text{then} & T = \mathbb{L}[X] \in \mathrm{Sym}_3^-\,. \end{array}\right\} \qquad (16)$$

We point out that in the isotropic case all the tensors X, Y, $X - Y$ and T are coaxial. Now let us consider separately the four cases $X \in W_k$ for $k = 0, 1, 2, 3$. For $T = \mathbb{L}[\mathbb{P}(X)] \in \mathrm{Sym}_k^-$, we firstly calculate the tensors $R_k(T)$ and $Q_k(T)$ that project the space of vectors onto its subspaces $\ker T$ and $\mathrm{ran}\, T$. Then, by using Proposition 3.2, we determine the tangent space $\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, \mathbb{L}^{-1}[T])$ and the normal space $\mathrm{Nor}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, \mathbb{L}^{-1}[T])$ to $\mathbb{L}^{-1}\,\mathrm{Sym}_k^-$ at $Y = \mathbb{L}^{-1}[T]$. We give the explicit expressions of the projections $\mathbb{Q}_k(Y)$ onto $\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y)$ and $\mathbb{R}_k(Y)$ onto $\mathrm{Nor}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y)$ and the fundamental form $\mathbb{B}_k(Y)$ is thus determined by using equation (9). Knowing the linear transformation $\mathbb{C}_k(Y, X - Y)$ from $\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_k^-, Y)$ into itself allows for calculating the derivative $\mathrm{D}\,\mathbb{P}(X)$ of $\mathbb{P}$ with respect to X, according to equation (11).

For $X \in W_0$ and then $T \in \mathrm{Sym}_0^-$, we have

$$R_0(T) = I,\ \ Q_0(T) = 0,$$

$$\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_0^-, Y) = \{0\}\,,$$

$$\mathrm{Nor}(\mathbb{L}^{-1}\,\mathrm{Sym}_0^-, Y) = \mathrm{Sym},$$

$$\mathbb{Q}_0(Y) = \mathbb{O},$$

$$\mathbb{R}_0(Y) = \mathbb{I},$$
$$\mathrm{D}\,\mathbb{P}(X) = \mathbb{O}.$$

For $X \in W_1$ and then $T \in \mathrm{Sym}_1^-$, it holds that

$$R_1(T) = I - O_{11},\ \ Q_1(T) = O_{11}, \tag{17}$$

$$\begin{aligned}\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_1^-, Y) &= \mathbb{L}^{-1}\,\mathrm{span}(O_{11}, O_{12}, O_{13})\\ &= \mathrm{span}(\mathbb{L}^{-1}[O_{11}], \mathbb{L}^{-1}[O_{12}], \mathbb{L}^{-1}[O_{13}]),\end{aligned}$$

$$\mathrm{Nor}(\mathbb{L}^{-1}\,\mathrm{Sym}_1^-, Y) = \{Z \in \mathrm{Sym}\ :\ Zq_1 = 0\} = \mathrm{span}(O_{22}, O_{33}, O_{23}),$$

The tensors

$$P_1 = \sqrt{E}\mathbb{L}^{-1}[O_{11}],\ \ P_2 = \sqrt{2\mu}\mathbb{L}^{-1}[O_{12}],\ \ P_3 = \sqrt{2\mu}\mathbb{L}^{-1}[O_{13}], \tag{18}$$

belonging to $\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_1^-, Y)$, and

$$P_4 = \frac{1}{\sqrt{2\mu+\lambda}}O_{22},\ \ P_5 = \frac{(2\mu+\lambda)O_{33} - \lambda O_{22}}{\sqrt{4\mu(\mu+\lambda)(2\mu+\lambda)}},\ \ P_6 = \frac{1}{\sqrt{2\mu}}O_{23},$$

belonging to $\mathrm{Nor}(\mathbb{L}^{-1}\,\mathrm{Sym}_1^-, Y)$, form a $\mathbb{L}$-orthonormal basis of Sym.

The projections $\mathbb{Q}_1(Y)$ and $\mathbb{R}_1(Y)$ respectively onto $\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_1^-, Y)$ and $\mathrm{Nor}(\mathbb{L}^{-1}\,\mathrm{Sym}_1^-, Y)$ are defined by

$$\mathbb{Q}_1(Y)[H] = (P_1, H)P_1 + (P_2, H)P_2 + (P_3, H)P_3, \qquad H \in \mathrm{Sym},$$

$$\mathbb{R}_1(Y)[H] = (P_4, H)P_4 + (P_5, H)P_5 + (P_6, H)P_6, \qquad H \in \mathrm{Sym}\,.$$

Recalling that $Y = \mathbb{P}(X)$, from $(14)_2$ we obtain

$$X - Y = \beta_2 O_{22} + \beta_3 O_{33} = p_4 P_4 + p_5 P_5,$$

where the coefficients β_2 and β_3 are

$$\beta_2 = \frac{\lambda x_1 + 2(\mu+\lambda)x_2}{2(\mu+\lambda)},\ \ \beta_3 = \frac{\lambda x_1 + 2(\mu+\lambda)x_3}{2(\mu+\lambda)}, \tag{19}$$

and p_4 and p_5 come from $(14)_2$ and (19)

$$p_4 = \frac{\lambda x_1 + (2\mu+\lambda)x_2 + \lambda x_3}{\sqrt{2\mu+\lambda}},$$

$$p_5 = (\lambda x_1 + 2(\mu+\lambda)x_3)\frac{\sqrt{\mu}}{\sqrt{(2\mu+\lambda)(\mu+\lambda)}}.$$

For $B, C \in \mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_1^-, Y)$ we put

$$S = R_1(T)(\mathbb{L}[C]T^{-1}\mathbb{L}[B] + \mathbb{L}[B]T^{-1}\mathbb{L}[C])R_1(T), \tag{20}$$

and from (9) we get

$$\begin{aligned}\mathbb{B}_1(Y)(B,C) &= \mathbb{R}_1(\mathbb{L}^{-1}[T])[\mathbb{L}^{-1}[S]]\\ &= (P_4\cdot S)P_4 + (P_5\cdot S)P_5 + (P_6\cdot S)P_6,\end{aligned}$$

and then

$$\begin{aligned}(X-Y,\mathbb{B}_1(Y)(B,C)) &= (X-Y)\cdot\mathbb{L}[\mathbb{B}_1(Y)(B,C)]\\ &= (p_4P_4+p_5P_5)\cdot\{(P_4\cdot S)\mathbb{L}[P_4]+(P_5\cdot S)\mathbb{L}[P_5]\\ &+(P_6\cdot S)\mathbb{L}[P_6]\}\\ &= p_4(P_4\cdot S)+p_5(P_5\cdot S).\end{aligned} \tag{21}$$

Having in mind that

$$T^{-1} = \frac{1}{Ex_1}O_{11},$$
$$P_4 = \frac{1}{\sqrt{2\mu+\lambda}}O_{22}$$

and

$$P_5 = \xi_2 O_{22} + \xi_3 O_{33},$$

with

$$\xi_2 = -\frac{\lambda}{2\sqrt{\mu(\mu+\lambda)(2\mu+\lambda)}},$$
$$\xi_3 = \frac{2\mu+\lambda}{2\sqrt{\mu(\mu+\lambda)(2\mu+\lambda)}},$$

from (20) and (17) we get

$$P_4\cdot S = \frac{2}{Ex_1}O_{11}\mathbb{L}[B]P_4\cdot\mathbb{L}[C],$$

$$P_5\cdot S = \frac{2}{Ex_1}O_{11}\mathbb{L}[B]P_5\cdot\mathbb{L}[C],$$

and finally, (21) becomes

$$\begin{aligned}(X\ -Y,\mathbb{B}_1(Y)(B,C)) &= \frac{2\mu p_4}{Ex_1\sqrt{2\mu+\lambda}}(\mathbb{L}[C]\cdot O_{12})(B\cdot O_{12})\\ &+\frac{2\mu p_5}{Ex_1}(\xi_2(\mathbb{L}[C]\cdot O_{12})(B\cdot O_{12})+\xi_3(\mathbb{L}[C]\cdot O_{13})(B\cdot O_{13})).\end{aligned} \tag{22}$$

From the relation

$$(\mathbb{C}_1(Y,X-Y)B,C) = (X-Y,\mathbb{B}_1(Y)(B,C))$$

for all $B,C\in \mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_1^-,Y)$, taking (22) into account, we obtain the explicit expression for $\mathbb{C}_1(Y,X-Y)$,

$$\mathbb{C}_1(Y,X-Y) = \frac{2\mu}{Ex_1}(\frac{p_4}{\sqrt{2\mu+\lambda}}+p_5\xi_2)P_2\otimes\mathbb{L}[P_2] + \frac{2\mu}{Ex_1}p_5\xi_2 P_3\otimes\mathbb{L}[P_3]. \tag{23}$$

Since the identity transformation on $\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_1^-, Y)$ is

$$\mathbb{I}_1(Y) = P_1 \otimes \mathbb{L}[P_1] + P_2 \otimes \mathbb{L}[P_2] + P_3 \otimes \mathbb{L}[P_3],$$

from (23), we get

$$\mathbb{I}_1(Y) - \mathbb{C}_1(Y, X-Y) = P_1 \otimes \mathbb{L}[P_1] + \frac{2\mu}{E}\frac{x_1 - x_2}{x_1} P_2 \otimes \mathbb{L}[P_2] + \frac{2\mu}{E}\frac{x_1 - x_3}{x_1} P_3 \otimes \mathbb{L}[P_3],$$

and relation (11) gives

$$\mathrm{D}\,\mathbb{P}(X) = P_1 \otimes \mathbb{L}[P_1] + \frac{E x_1}{2\mu(x_1 - x_2)} P_2 \otimes \mathbb{L}[P_2] + \frac{E x_1}{2\mu(x_1 - x_3)} P_3 \otimes \mathbb{L}[P_3]. \quad (24)$$

By using the expressions for the derivative of eigenvalues and eigenvectors of a symmetric tensor summarized in [16], the derivative of $\mathbb{P}(X)$ in $(14)_2$ turns out to be

$$\begin{aligned} \mathrm{D}\,\mathbb{P}(X) \;&= \frac{2+3\alpha}{2(1+\alpha)} O_{11} \otimes O_{11} + \frac{(2+3\alpha)x_1}{2(1+\alpha)} \Big(\frac{1}{x_1 - x_2} O_{12} \otimes O_{12} \\ &+ \frac{1}{x_1 - x_3} O_{13} \otimes O_{13} \Big) - \frac{\alpha}{2(1+\alpha)} I \otimes O_{11}. \end{aligned} \quad (25)$$

Having in mind expressions (18) linking P_1, P_2, P_3 and O_{11}, O_{12}, O_{13}, it is easy to verify that (24) and (25) coincide.

For $X \in W_2$ and then $T \in \mathrm{Sym}_2^-$, it holds that

$$R_2(T) = O_{33}, \;\; Q_2(T) = I - O_{33}, \quad (26)$$

$$\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_2^-, Y) = \mathbb{L}^{-1}(\mathrm{span}(O_{33})^{\perp}),$$

$$\mathrm{Nor}(\mathbb{L}^{-1}\,\mathrm{Sym}_2^-, Y) = \{Z \in \mathrm{Sym} \,:\, Zq_1 = Zq_2 = 0\} = \mathrm{span}(O_{33}),$$

The tensors

$$P_1 = \sqrt{E}\mathbb{L}^{-1}[O_{11}], \;\; P_2 = \sqrt{2\mu}\mathbb{L}^{-1}[O_{12}], \;\; P_3 = \sqrt{2\mu}\mathbb{L}^{-1}[O_{13}],$$

$$P_4 = \sqrt{2\mu}\mathbb{L}^{-1}[O_{213}], \;\; P_5 = 2\sqrt{\frac{\mu(\mu+\lambda)}{2\mu+\lambda}} \left(\frac{\lambda}{2(\mu+\lambda)} \mathbb{L}^{-1}[O_{11}] + \mathbb{L}^{-1}[O_{22}] \right),$$

belonging to $\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_2^-, Y)$, and

$$P_6 = \frac{1}{\sqrt{2\mu+\lambda}} O_{33},$$

in $\mathrm{Nor}(\mathbb{L}^{-1}\,\mathrm{Sym}_2^-, Y)$ form a $\mathbb{L}$-orthonormal basis of Sym.

The projections $\mathbb{Q}_2(Y)$ and $\mathbb{R}_2(Y)$ respectively onto $\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_2^-, Y)$ and $\mathrm{Nor}(\mathbb{L}^{-1}\,\mathrm{Sym}_2^-, Y)$ are defined by

$$\begin{aligned} \mathbb{Q}_2(Y)[H] \;&= (P_1, H)P_1 + (P_2, H)P_2 + (P_3, H)P_3 \\ &+ (P_4, H)P_4 + (P_5, H)P_5, \qquad H \in \mathrm{Sym}, \end{aligned} \quad (27)$$

$$\mathbb{R}_2(Y)[H] = (P_6, H)P_6, \qquad H \in \mathrm{Sym}\,.$$

Having in mind the expression of $Y = \mathbb{P}(X)$ in $(14)_3$, we have

$$X - Y = \beta_3 O_{33} = p_6 P_6,$$

with

$$\beta_3 = \frac{(2\mu+\lambda)x_3 + \lambda(x_1+x_2)}{2\mu+\lambda},$$
$$p_6 = \frac{(2\mu+\lambda)x_3 + \lambda(x_1+x_2)}{\sqrt{2\mu+\lambda}},$$

For $B, C \in \mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_2^-, Y)$, putting

$$S = R_2(T)(\mathbb{L}[C]T^{-1}\mathbb{L}[B] + \mathbb{L}[B]T^{-1}\mathbb{L}[C])R_2(T), \tag{28}$$

we have

$$\mathbb{B}_2(Y)(B, C) = \mathbb{R}_2(Y)[\mathbb{L}^{-1}[S]] = (P_6 \cdot S)P_6, \tag{29}$$

and then

$$\begin{aligned}(X - Y, \mathbb{B}_2(Y)(B, C)) &= (X - Y) \cdot \mathbb{L}[\mathbb{B}_2(Y)(B, C)] \\ &= p_6 P_6 \cdot (P_6 \cdot S)\mathbb{L}[P_6] = p_6(P_6 \cdot S).\end{aligned} \tag{30}$$

From $(16)_3$ it follows that

$$T^{-1} = \frac{2\mu+\lambda}{2\mu}\left\{\frac{1}{2(\mu+\lambda)x_1 + \lambda x_2}O_{11} + \frac{1}{2(\mu+\lambda)x_2 + \lambda x_1}O_{22}\right\},$$

and from (28) and (26) we get

$$\begin{aligned}&(X - Y, \mathbb{B}_2(Y)(B, C)) \\ &\quad = \frac{2\mu p_6}{\sqrt{2\mu+\lambda}}\frac{2\mu+\lambda}{2\mu}\frac{1}{2(\mu+\lambda)x_1+\lambda x_2}(\mathbb{L}[C]\cdot O_{13})(B\cdot O_{13}) \\ &\quad + \frac{2\mu p_6}{\sqrt{2\mu+\lambda}}\frac{2\mu+\lambda}{2\mu}\frac{1}{2(\mu+\lambda)x_2+\lambda x_1}(\mathbb{L}[C]\cdot O_{23})(B\cdot O_{23})\end{aligned} \tag{31}$$

From the relation

$$(\mathbb{C}_2(Y, X - Y)B, C) = (X - Y, \mathbb{B}_2(Y)(B, C))$$

for all $B, C \in \mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_2^-, Y)$, taking (31) into account, we obtain the explicit expression for $\mathbb{C}_2(Y, X - Y)$,

$$\begin{aligned}\mathbb{C}_2(Y, X - Y) &= \frac{(2\mu+\lambda)x_3 + \lambda(x_1+x_2)}{2(\mu+\lambda)x_1 + \lambda x_2}P_3 \otimes \mathbb{L}[P_3] \\ &= \frac{(2\mu+\lambda)x_3 + \lambda(x_1+x_2)}{\lambda x_1 + 2(\mu+\lambda)x_2}P_4 \otimes \mathbb{L}[P_4].\end{aligned} \tag{32}$$

Thus, for

$$\mathbb{I}_2(Y) = P_1 \otimes \mathbb{L}[P_1] + P_2 \otimes \mathbb{L}[P_2] + P_3 \otimes \mathbb{L}[P_3] + P_4 \otimes \mathbb{L}[P_4] + P_5 \otimes \mathbb{L}[P_5], \quad (33)$$

the identity transformation on $\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_2^-, Y)$, from (32), we have

$$\begin{aligned}\mathbb{I}_2(Y) - &\mathbb{C}_2(Y, X - Y) \\ &= P_1 \otimes \mathbb{L}[P_1] + P_2 \otimes \mathbb{L}[P_2] + \frac{(2\mu+\lambda)(x_1 - x_3)}{2(\mu+\lambda)x_1 + \lambda x_2} P_3 \otimes \mathbb{L}[P_3] \\ &+ \frac{(2\mu+\lambda)(x_2 - x_3)}{\lambda x_1 + 2(\mu+\lambda)x_2} P_4 \otimes \mathbb{L}[P_4] + P_5 \otimes \mathbb{L}[P_5],\end{aligned}$$

and relation (11) gives

$$\mathrm{D}\,\mathbb{P}(X) = P_1 \otimes \mathbb{L}[P_1] + P_2 \otimes \mathbb{L}[P_2] + \frac{2(\mu+\lambda)x_1 + \lambda x_2}{(2\mu+\lambda)(x_1 - x_3)} P_3 \otimes \mathbb{L}[P_3] +$$

$$\frac{\lambda x_1 + 2(\mu+\lambda)x_2}{(2\mu+\lambda)(x_2 - x_3)} P_4 \otimes \mathbb{L}[P_4] + P_5 \otimes \mathbb{L}[P_5],$$

which coincides with the derivative of $\mathbb{P}(X)$ with respect to X calculated by differentiating $(14)_3$ and using the expressions of the derivative of eigenvalues and eigenvectors of X [16].

Finally, if $X \in W_3$ then $T \in \mathrm{Sym}_3^-$ and we have

$$R_3(T) = 0, \quad Q_3(T) = \mathrm{I},$$

$$\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_3^-, Y) = \mathrm{Sym},$$

$$\mathrm{Nor}(\mathbb{L}^{-1}\,\mathrm{Sym}_3^-, Y) = \{0\}\,.$$

$$\mathbb{Q}_3(Y) = \mathrm{I},$$

$$\mathbb{R}_3(Y) = \mathbb{O},$$

and

$$\mathrm{D}\,\mathbb{P}(X) = \mathbb{I}.$$

When the derivative $\mathrm{D}\,\mathbb{P}(X)$ is known, the derivative of the stress $T = \mathbb{L}[\mathbb{P}(X)]$ with respect to X is $\mathrm{D}\,\mathbb{T}(X) = \mathbb{L}\,\mathrm{D}\,\mathbb{P}(X)$, and, in particular,

$$\mathrm{D}\,\mathbb{T}(X) = 2\mu\,\mathrm{D}\,\mathbb{P}(X) + \lambda \mathrm{I} \otimes \mathrm{D}\,\mathbb{P}(X)^T[\mathrm{I}],$$

where $\mathrm{D}\,\mathbb{P}(X)^T$ is the transpose of the fourth-order tensor $\mathrm{D}\,\mathbb{P}(X)$ defined by

$$\mathrm{D}\,\mathbb{P}(X)^T[H] \cdot K = \mathrm{D}\,\mathbb{P}(X)[K] \cdot H, \quad \text{for every } H, K \in \mathrm{Sym}\,.$$

7 The anisotropic case

In principle for $\mathbb{L}$ anisotropic the derivative $\mathrm{D}\,\mathbb{P}(X)$ can be calculated by following the same procedure of the isotropic case. For example, let us consider the case

$$X \in W_1 = \text{interior of } \left\{ A \in \mathrm{Sym} \,:\, \mathbb{L}[\mathbb{P}(A)] \in \mathrm{Sym}_1^- \right\},$$

then $T = t_1 O_{11}$, where $t_1 < 0$ and $O_{11} = q_1 \otimes q_1$, (see (13)) with q_1, q_2, q_3 the eigenvectors of T. Here, unlike the isotropic case, the vectors q_1, q_2, q_3 are not eigenvectors of X and their dependence on X is unknown, since, at the moment, the explicit expression of the projection $Y = \mathbb{P}(X)$ is not available. On the other hand, q_1, q_2, q_3 are eigenvector of $X - Y$ and

$$X - Y = a_2 O_{22} + a_3 O_{33},$$

with $a_2, a_3 \geq 0$. The calculation of $\mathrm{D}\,\mathbb{P}(X)$ requires the following steps.

Step 1 Determine the tensors

$$R_1(T) = I - O_{11}, Q_1(T) = O_{11},$$

and the subspaces

$$\begin{aligned}\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_1^-, Y) &= \mathbb{L}^{-1}\,\mathrm{span}(O_{11}, O_{12}, O_{13}) \\ &= \mathrm{span}(\mathbb{L}^{-1}[O_{11}], \mathbb{L}^{-1}[O_{12}], \mathbb{L}^{-1}[O_{13}]),\end{aligned}$$

$$\mathrm{Nor}(\mathbb{L}^{-1}\,\mathrm{Sym}_1^-, Y) = \{Z \in \mathrm{Sym} \,:\, Zq_1 = 0\} = \mathrm{span}(O_{22}, O_{33}, O_{23}).$$

Step 2 Determine a $\mathbb{L}$-orthonormal basis P_i, $i = 1, ..., 6$ of Sym such that

$$P_1,\ P_2,\ P_3 \in \mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_1^-, Y),$$

$$P_4,\ P_5,\ P_6 \in \mathrm{Nor}(\mathbb{L}^{-1}\,\mathrm{Sym}_1^-, Y).$$

In particular,

$$\begin{aligned} P_i &= \xi_{11}^{(i)}\mathbb{L}^{-1}[O_{11}] + \xi_{12}^{(i)}\mathbb{L}^{-1}[O_{12}] + \xi_{13}^{(i)}\mathbb{L}^{-1}[O_{13}],\ i = 1, 2, 3, \\ P_i &= \xi_{22}^{(i)} O_{22} + \xi_{33}^{(i)} O_{33} + \xi_{23}^{(i)} O_{23},\ i = 4, 5, 6, \end{aligned} \tag{34}$$

with $\xi_{kl}^{(i)} \in \mathbb{R}$.

Step 3 Determine the projectors

$$\mathbb{Q}_1(Y) = P_1 \otimes \mathbb{L}[P_1] + P_2 \otimes \mathbb{L}[P_2] + P_3 \otimes \mathbb{L}[P_3],$$

$$\mathbb{R}_1(Y) = P_4 \otimes \mathbb{L}[P_4] + P_5 \otimes \mathbb{L}[P_5] + P_6 \otimes \mathbb{L}[P_6].$$

Step 4 Considering that

$$X - Y = b_4 P_4 + b_5 P_5 + b_6 P_6,$$

we get

$$\mathbb{C}_1(Y, X-Y) = \sum_{j=4,5,6} \frac{b_j}{t_1} \Big\{ \xi_{22}^{(j)} O_{12} \otimes \mathbb{L}[O_{12}] + \xi_{33}^{(j)} O_{13} \otimes \mathbb{L}[O_{13}]$$

$$+ 2^{-1/2} \xi_{23}^{(j)} (O_{12} \otimes \mathbb{L}[O_{13}] + O_{13} \otimes \mathbb{L}[O_{12}]) \Big\}.$$

Step 5 For

$$\mathbb{I}_1(Y) = P_1 \otimes \mathbb{L}[P_1] + P_2 \otimes \mathbb{L}[P_2] + P_3 \otimes \mathbb{L}[P_3],$$

the identity transformation on $\mathrm{Tan}(\mathbb{L}^{-1}\,\mathrm{Sym}_1^-, Y)$, find the spectral decomposition of $\mathbb{I}_1(Y) - \mathbb{C}_1(Y, X-Y)$ and then determine $(\mathbb{I}_1(Y) - \mathbb{C}_1(Y, X-Y))^{-1}$. The expression for $\mathrm{D}\,\mathbb{P}(X)$ comes from (11). Such a procedure could be implemented in the NOSA-ITACA code and applied to solve the equilibrium problem of anisotropic no-tension solids. In particular, once $Y = \mathbb{P}(X)$, and then $T = \mathbb{L}[Y]$, is calculated numerically, its derivative can be calculated numerically as well, by following steps 1–5, thus allowing determination of the tangent stiffness matrix.

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