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Defeasible RDFS via rational closure

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Keywords: RDFS, Non-monotone logics, Rational Closure.

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Defeasible RDFS via Rational Closure

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Abstract

In the field of non-monotonic logics, the notion of Rational Closure (RC) is acknowledged as a prominent approach. In recent years, RC has gained even more popularity in the context of Description Logics (DLs), the logic underpinning the semantic web standard ontology language OWL 2, whose main ingredients are classes and roles.

In this work, we show how to integrate RC within the triple language RDFS, which together with OWL 2 are the two major standard semantic web ontology languages. To do so, we start from ρdf, which is the logic behind RDFS, and then extend it to $\rho df_\bot$, allowing to state that two entities are incompatible. Eventually, we propose defeasible $\rho df_\bot$ via a typical RC construction.

The main features of our approach are: (i) unlike most other approaches that add an extra non-monotone rule layer on top of monotone RDFS, defeasible $\rho df_\bot$ remains syntactically a triple language and is a simple extension of ρdf by introducing some new predicate symbols with specific semantics. In particular, any RDFS reasoner/store may handle them as ordinary terms if it does not want to take account for the extra semantics of the new predicate symbols; (ii) the defeasible $\rho df_\bot$ entailment decision procedure is build on top of the $\rho df_\bot$ entailment decision procedure, which in turn is an extension of the one for ρdf via some additional inference rules favouring an potential implementation; and (iii) defeasible $\rho df_\bot$ entailment can be decided in polynomial time.

1 Introduction

Description Logics [13] (DLs) under *Rational Closure* [58] (RC) is a well-known framework for non-monotonic reasoning in DLs, which has gained rising attention in the last decade (see, *e.g.*, [22, 23, 24, 25, 26, 29, 27, 30, 38, 40, 41, 42, 46, 45, 43, 44, 47, 20, 74, 70, 69, 71] for related approaches).

We recall that a typical problem that can be addressed using non-monotonic formalisms is reasoning with ontologies in which some classes are exceptional w.r.t. some properties of their super classes, as illustrated with the well-known "penguin example".

Example 1.1. *We know that penguins are birds, birds usually fly, while penguins do not.* □

While DLs provide the logical foundation of formal ontologies of the OWL family [66] and endowing them with non-monotonic features is still a main issue, as documented by the past 20 years of technical development (see *e.g.*, [19, 35, 34, 39, 62] and references therein), addressing non-monotonicity in the context of the triple language RDFS [76], which together with OWL 2 are the two major standard semantic web ontology languages, has attracted in comparison little attention so far. Almost all approaches we are aware of consider a so-called rule-layer on top of RDFS (see *e.g.*, [7, 8, 36, 35, 53] and Section 4).

In this paper, we will show how to integrate RC within the triple language RDFS. To to do so, we start from ρdf [63], which is the logic behind RDFS, and then extend it to $\rho df_\bot$, allowing to state that two entities are incompatible. So, for instance, by referring to Example 1.1, we may represent that flying creatures (f) and non-flying creatures (e) are incompatible each other via the $\rho df_\bot$ triple $(e, \bot_{\mathsf{c}}, f)$ (see also Example 3.1 later on), where $\bot_{\mathsf{c}}$ is a new predicate added to the ρdf vocabulary to model an incompatibility relation. Based on $\rho df_\bot$, we will then propose defeasible $\rho df_\bot$ via a typical RC construction, allowing to state, *e.g.*, "birds (b) usually fly" via the defeasible triple $\langle b, \mathsf{sc}, f\rangle$, alongside classical triples such as (p, sc, b) ("penguins (p) are birds") and (p, sc, e) ("penguins are non-flying creatures").[1] To the best of our knowledge, this is the first attempt to include RC within RDFS.

The main features of our approach are:

- unlike other approaches that add an extra non-monotone rule layer on top of monotone RDFS, defeasible $\rho df_\bot$ remains syntactically a triple language and is a simple extension of ρdf by introducing some new predicate symbols with specific semantics. In particular, any RDFS reasoner/store and SPARQL [50] query answering tool may handle them as ordinary terms if it does not want to take account for the extra semantics of the new predicate symbols added to ρdf;
- the defeasible $\rho df_\bot$ entailment decision procedure is build on top of the $\rho df_\bot$ entailment decision procedure, which in turn is an extension of the one for ρdf via some additional inference rules favouring an potential implementation; and
- defeasible $\rho df_\bot$ entailment can be decided in polynomial time.

In the following, we will proceed as follows. In the next section, we will introduce $\rho df_\bot$, by defining its syntax, semantics and entailment decision procedure. In Section 3 we extend $\rho df_\bot$ towards defeasible $\rho df_\bot$ via a RC construction, defining syntax, semantics, entailment decision procedure and address its computational complexity. Section 4 concludes, summarises related work and highlights future work.

2 $\rho df_\bot$ graphs

RC is one of the most popular non-monotonic approaches in conditional reasoning. Unlike other approaches to non-monotonic reasoning for RDFS (see Section 4), that are build using some form of *negation as failure*, conditional reasoning systems are usually based on a different technical solution: given a set of defeasible conditionals $\alpha \rightsquigarrow \beta$ (if α holds, then presumably β holds), we use them like classical monotonic conditionals until we have a conflict in our information, that triggers the non-monotonic reasoning machinery. Example 1.1 is a classical example: we have a conflict between classical reasoning, suggesting that penguins, being birds, should fly, and exceptional more specific information, stating that penguins do not fly. Faced with such a conflict, the non-monotonic engine solves it giving, in this case, precedence to the more specific information (penguins do not fly). This kind of reasoning allows a good level of flexibility: we can freely add in our premises exceptional cases that are in conflict with the general rules (that is, the conditionals), and the non-monotonic machinery will be able to autonomously manage such conflicts. However, in order to implement this kind of defeasible reasoning in the RDFS framework, we need to introduce some notion of informational incompatibility. In the following, we extend the ρdf language to the $\rho df_\bot$ language allowing to express some form of conflicts.

[1] According to ρdf, sc stands for "is subclass of".

2.1 Syntax

We rely on a fragment of RDFS, called *minimal* ρdf [63, Def. 15], that covers essential features of RDFS. Specifically, minimal ρdf is defined as the following subset of the RDFS vocabulary:

$$\rho\text{df} = \{\mathsf{sp}, \mathsf{sc}, \mathsf{type}, \mathsf{dom}, \mathsf{range}\} \ .$$

Moreover, it does not consider so-called *blank* nodes and, thus, in what follows, triples and graphs will be *ground*. In fact, minimal ρdf suffices to illustrate the main concepts and algorithms we will consider in this work and ease the presentation. To avoid unnecessary redundancy, we will just drop the term 'minimal' in what follows.

So, consider pairwise disjoint alphabets $\mathbf{U}$ and $\mathbf{L}$ denoting, respectively, *URI references* and *literals*.[2] We assume that $\mathbf{U}$ contains the ρdf vocabulary. A literal may be a *plain literal* (*e.g.*, a string) or a *typed literal* (*e.g.*, a boolean value) [61]. We call the elements in $\mathbf{UL}$ *terms*. Terms are denoted with lower case letters $a, b, \ldots$ with optional super/lower script.

A *triple* is of the form $\tau = (s, p, o) \in \mathbf{UL} \times \mathbf{U} \times \mathbf{UL}$,[3] where $s, o \notin \rho$df. We call s the *subject*, p the *predicate*, and o the *object*.

A *graph* G is a set of triples, the *universe* of G, denoted $\mathtt{uni}(G)$, is the set of terms in $\mathbf{UL}$ that occur in the triples of G.

We recall that informally (i) (p, sp, q) means that property p is a *subproperty* of property q; (ii) (c, sc, d) means that class c is a *subclass* of class d; (iii) (a, type, b) means that a is of *type* b; (iv) (p, dom, c) means that the *domain* of property p is c; and (v) (p, range, c) means that the *range* of property p is c.

We extend the vocabulary of ρdf with a new pair of predicates, $\bot_{\mathsf{c}}$ and $\bot_{\mathsf{p}}$, representing incompatible information: (vi) $(c, \bot_{\mathsf{c}}, d)$ indicates that the classes c and d are disjoint and, analogously, (vii) $(p, \bot_{\mathsf{p}}, q)$ indicates that the properties p and q are disjoint.

We call $\rho df_\bot$ the vocabulary obtained from ρdf by adding $\bot_{\mathsf{c}}$ and $\bot_{\mathsf{p}}$, that is,

$$\rho df_\bot = \{\mathsf{sp}, \mathsf{sc}, \mathsf{type}, \mathsf{dom}, \mathsf{range}, \bot_{\mathsf{c}}, \bot_{\mathsf{p}}\} \ .$$

Like for ρdf, we assume that $\mathbf{U}$ contains the $\rho df_\bot$ vocabulary and that all triples $(s, p, o) \in \mathbf{UL} \times \mathbf{U} \times \mathbf{UL}$ are such that $s, o \notin \rho df_\bot$.

Remark 2.1. *Please, note that we allow the $\rho df_\bot$ predicates to occur only as second elements of the triples, that is, we allow triples (p, sp, q), but not triples such as e.g., (sp, p, o) or $(\bot_{\mathsf{p}}, p, o)$, which is in line with the notion of* minimal ρdf triple *[63, Def. 15].*

2.2 Semantics

An *interpretation* $\mathcal{I}$ over a vocabulary V is a tuple $\mathcal{I} = \langle \Delta_{\mathsf{R}}, \Delta_{\mathsf{P}}, \Delta_{\mathsf{C}}, \Delta_{\mathsf{L}}, \mathfrak{P}[\![\cdot]\!], \mathfrak{C}[\![\cdot]\!], \cdot^{\mathcal{I}} \rangle$, where $\Delta_{\mathsf{R}}, \Delta_{\mathsf{P}}, \Delta_{\mathsf{C}}, \Delta_{\mathsf{L}}$ are the interpretation domains of $\mathcal{I}$, which are finite non-empty sets, and $\mathfrak{P}[\![\cdot]\!], \mathfrak{C}[\![\cdot]\!], \cdot^{\mathcal{I}}$ are the interpretation functions of $\mathcal{I}$. In particular:

1. Δ_{R} are the resources (the domain or universe of $\mathcal{I}$);
2. Δ_{P} are property names (not necessarily disjoint from Δ_{R});
3. $\Delta_{\mathsf{C}} \subseteq \Delta_{\mathsf{R}}$ are the classes;
4. $\Delta_{\mathsf{L}} \subseteq \Delta_{\mathsf{R}}$ are the literal values and contains $\mathbf{L} \cap V$;

[2] We assume $\mathbf{U}$, and $\mathbf{L}$ fixed, and for ease we will denote unions of these sets simply concatenating their names.

[3] As in [63] we allow literals for s.

5. $\mathfrak{P}[\![\cdot]\!]$ is a function $\mathfrak{P}[\![\cdot]\!] : \Delta_{\mathsf{P}} \to 2^{\Delta_{\mathsf{R}} \times \Delta_{\mathsf{R}}}$;
6. $\mathfrak{C}[\![\cdot]\!]$ is a function $\mathfrak{C}[\![\cdot]\!] : \Delta_{\mathsf{C}} \to 2^{\Delta_{\mathsf{R}}}$;
7. $\cdot^{\mathcal{I}}$ maps each $t \in \mathbf{UL} \cap V$ into a value $t^{\mathcal{I}} \in \Delta_{\mathsf{R}} \cup \Delta_{\mathsf{P}}$, and such that $\cdot^{\mathcal{I}}$ is the identity for plain literals and assigns an element in Δ_{R} to each element in $\mathbf{L}$.

Definition 2.1 (Satisfaction). *An interpretation $\mathcal{I}$ is a* model *of a graph G, denoted $\mathcal{I} \Vdash_{\rho df_\bot} G$, if and only if $\mathcal{I}$ is an interpretation over the vocabulary $\rho df_\bot \cup \mathtt{uni}(G)$ that satisfies the following conditions:*

Simple:

1. *for each $(s, p, o) \in G$, $p^{\mathcal{I}} \in \Delta_{\mathsf{P}}$ and $(s^{\mathcal{I}}, o^{\mathcal{I}}) \in \mathfrak{P}[\![p^{\mathcal{I}}]\!]$;*

Subproperty:

1. *$\mathfrak{P}[\![\mathsf{sp}^{\mathcal{I}}]\!]$ is transitive over Δ_{P};*
2. *if $(p, q) \in \mathfrak{P}[\![\mathsf{sp}^{\mathcal{I}}]\!]$ then $p, q \in \Delta_{\mathsf{P}}$ and $\mathfrak{P}[\![p]\!] \subseteq \mathfrak{P}[\![q]\!]$;*

Subclass:

1. *$\mathfrak{P}[\![\mathsf{sc}^{\mathcal{I}}]\!]$ is transitive over Δ_{C};*
2. *if $(c, d) \in \mathfrak{P}[\![\mathsf{sc}^{\mathcal{I}}]\!]$ then $c, d \in \Delta_{\mathsf{C}}$ and $\mathfrak{C}[\![c]\!] \subseteq \mathfrak{C}[\![d]\!]$;*

Typing I:

1. *$x \in \mathfrak{C}[\![c]\!]$ if and only if $(x, c) \in \mathfrak{P}[\![\mathsf{type}^{\mathcal{I}}]\!]$;*
2. *if $(p, c) \in \mathfrak{P}[\![\mathsf{dom}^{\mathcal{I}}]\!]$ and $(x, y) \in \mathfrak{P}[\![p]\!]$ then $x \in \mathfrak{C}[\![c]\!]$;*
3. *if $(p, c) \in \mathfrak{P}[\![\mathsf{range}^{\mathcal{I}}]\!]$ and $(x, y) \in \mathfrak{P}[\![p]\!]$ then $y \in \mathfrak{C}[\![c]\!]$;*

Typing II:

1. *For each $\mathsf{e} \in \rho df_\bot$, $\mathsf{e}^{\mathcal{I}} \in \Delta_{\mathsf{P}}$*
2. *if $(p, c) \in \mathfrak{P}[\![\mathsf{dom}^{\mathcal{I}}]\!]$ then $p \in \Delta_{\mathsf{P}}$ and $c \in \Delta_{\mathsf{C}}$*
3. *if $(p, c) \in \mathfrak{P}[\![\mathsf{range}^{\mathcal{I}}]\!]$ then $p \in \Delta_{\mathsf{P}}$ and $c \in \Delta_{\mathsf{C}}$*
4. *if $(x, c) \in \mathfrak{P}[\![\mathsf{type}^{\mathcal{I}}]\!]$ then $c \in \Delta_{\mathsf{C}}$*

Disjointness I:

1. *if $(c, d) \in \mathfrak{P}[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$ then $c, d \in \Delta_{\mathsf{C}}$;*
2. *$\mathfrak{P}[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$ is symmetric,* sc*-transitive and* c*-exhaustive over Δ_{C} (see below);*
3. *if $(p, q) \in \mathfrak{P}[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$ then $p, q \in \Delta_{\mathsf{P}}$;*
4. *$\mathfrak{P}[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$ is symmetric,* sp*-transitive and* p*-exhaustive over Δ_{P} (see below);*

Disjointness II:

1. *If $(p, c) \in \mathfrak{P}[\![\mathsf{dom}^{\mathcal{I}}]\!]$, $(q, d) \in \mathfrak{P}[\![\mathsf{dom}^{\mathcal{I}}]\!]$, and $(c, d) \in \mathfrak{P}[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$, then $(p, q) \in \mathfrak{P}[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$;*
2. *If $(p, c) \in \mathfrak{P}[\![\mathsf{range}^{\mathcal{I}}]\!]$, $(q, d) \in \mathfrak{P}[\![\mathsf{range}^{\mathcal{I}}]\!]$, and $(c, d) \in \mathfrak{P}[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$, then $(p, q) \in \mathfrak{P}[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$.*

Symmetry:

1. *If $(c, d) \in \mathfrak{P}[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$, then $(d, c) \in \mathfrak{P}[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$*

2. *If* $(p,q) \in \mathfrak{P}[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$*, then* $(q,p) \in \mathfrak{P}[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$

sc-Transitivity:

1. *If* $(c,d) \in \mathfrak{P}[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$ *and* $(e,c) \in \mathfrak{P}[\![\mathsf{sc}^{\mathcal{I}}]\!]$*, then* $(e,d) \in \mathfrak{P}[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$

sp-Transitivity:

1. *If* $(p,q) \in \mathfrak{P}[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$ *and* $(r,p) \in \mathfrak{P}[\![\mathsf{sp}^{\mathcal{I}}]\!]$*, then* $(r,q) \in \mathfrak{P}[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$

c-Exhaustive:

1. *If* $(c,c) \in \mathfrak{P}[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$ *and* $d \in \Delta_{\mathsf{C}}$ *then* $(c,d) \in \mathfrak{P}[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$*;*

p-Exhaustive:

1. *If* $(p,p) \in \mathfrak{P}[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$ *and* $q \in \Delta_{\mathsf{P}}$ *then* $(p,q) \in \mathfrak{P}[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$.

A graph G *is* satisfiable *if it has a model* $\mathcal{I}$ *(denoted* $\mathcal{I} \Vdash_{\rho df_\bot} G$*).*

Remark 2.2. *Please note that we have built our semantics starting from the so-called* reflexive-relaxed *ρdf semantics [63], in which the predicates* sc *and* sp *are* not *assumed to be reflexive in accordance to the notion of minimal graph [63, Def. 15]. Our additional semantic constraints w.r.t. [63, Def. 15] are those of condition **Disjointness I** onwards.*

Remark 2.3. *Note that the presence of e.g.,* $(a, \mathsf{type}, b), (a, \mathsf{type}, c)$ *and* $(b, \bot_{\mathsf{c}}, c)$ *in a graph does not preclude its satisfiability. In fact, a graph is always satisfiable (see Corollary 2.5 later on) avoiding, thus, the possibility of unsatisfiability and the* ex falso quodlibet *principle, which is in line with the ρdf semantics [64, 63].*

On top of the notion of satisfaction we define a notion of entailment between graphs.

Definition 2.2 (Entailment $\models_{\rho df_\bot}$)**.** *Given two graphs* G *and* H*, we say that* G entails H*, denoted* $G \models_{\rho df_\bot} H$*, if and only if every model of* G *is also a model of* H*.*

2.3 Deductive system

In what follows, we provide a sound and complete deductive system for our language. Our system extends the classical *minimal* ρdf system as by [63, Proposition 17]. The system is arranged in groups of rules that capture the semantic conditions of models. In every rule, A, B, C, D, E, X, and Y are meta-variables representing elements in $\mathbf{UL}$. An instantiation of a rule is obtained by replacing those meta-variables with actual terms.

The rules are as follows:

1. Simple:

   $\frac{G}{G'}$ for $G' \subseteq G$

2. Subproperty:

   (a) $\frac{(A,\mathsf{sp},B),(B,\mathsf{sp},C)}{(A,\mathsf{sp},C)}$ (b) $\frac{(D,\mathsf{sp},E),(X,D,Y)}{(X,E,Y)}$

3. Subclass:

   (a) $\frac{(A,\mathsf{sc},B),(B,\mathsf{sc},C)}{(A,\mathsf{sc},C)}$ (b) $\frac{(A,\mathsf{sc},B),(X,\mathsf{type},A)}{(X,\mathsf{type},B)}$

4. Typing:

(a) $\frac{(D,\mathsf{dom},B),(X,D,Y)}{(X,\mathsf{type},B)}$ (b) $\frac{(D,\mathsf{range},B),(X,D,Y)}{(Y,\mathsf{type},B)}$

5. Conceptual Disjointness:

(a) $\frac{(A,\bot_{\mathsf{c}},B)}{(B,\bot_{\mathsf{c}},A)}$ (b) $\frac{(A,\bot_{\mathsf{c}},B),(C,\mathsf{sc},A)}{(C,\bot_{\mathsf{c}},B)}$ (c) $\frac{(A,\bot_{\mathsf{c}},A)}{(A,\bot_{\mathsf{c}},B)}$

6. Predicate Disjointness:

(a) $\frac{(A,\bot_{\mathsf{p}},B)}{(B,\bot_{\mathsf{p}},A)}$ (b) $\frac{(A,\bot_{\mathsf{p}},B),(C,\mathsf{sp},A)}{(C,\bot_{\mathsf{p}},B)}$ (c) $\frac{(A,\bot_{\mathsf{p}},A)}{(A,\bot_{\mathsf{p}},B)}$

7. Crossed Disjointness:

(a) $\frac{(A,\mathsf{dom},C),(B,\mathsf{dom},D),(C,\bot_{\mathsf{c}},D)}{(A,\bot_{\mathsf{p}},B)}$ (b) $\frac{(A,\mathsf{range},C),(B,\mathsf{range},D),(C,\bot_{\mathsf{c}},D)}{(A,\bot_{\mathsf{p}},B)}$

Please note that the rules that extend minimal ρdf to $\rho df_\bot$ are rules (5) - (7).

Now, using this rules we define a derivation relation in a similar way as in [63, 64].

Definition 2.3 (Derivation $\vdash_{\rho df_\bot}$)**.** *Let G and H be $\rho df_\bot$-graphs. $G \vdash_{\rho df_\bot} H$ iff there exists a sequence of graphs $P_1, P_2, \ldots, P_k$ with $P_1 = G$ and $P_k = H$ and for each j ($2 \leqslant j \leqslant k$) one of the following cases hold:*

- *$P_j \subseteq P_{j-1}$ (rule (1));*
- *there is an instantiation R/R' of one of the rules (2)-(7), such that $R \subseteq P_{j-1}$ and $P_j = P_{j-1} \cup R'$.*

Such sequence of graphs is called a proof of $G \vdash_{\rho df_\bot} H$. Whenever $G \vdash_{\rho df_\bot} H$, we say that the graph H is derived from the graph G. Each pair (P_{j-1}, P_j), $1 \leqslant j \leqslant k$ is called a step of the proof which is labeled by the respective instantiation R/R' of the rule applied at the step.

We are going now to prove the following soundness and completeness theorem.

Theorem 2.1 (Soundness & Completeness)**.** *Let G and H be $\rho df_\bot$-graphs.*

$$G \vdash_{\rho df_\bot} H \textit{ iff } G \models_{\rho df_\bot} H\ .$$

We divide the proof into lemmas. The following one is needed for soundness.

Lemma 2.2. *Let G and H be $\rho df_\bot$-graphs, let G be satisfiable, and let one of the following statements hold:*

- *$H \subseteq G$;*
- *there is an instantiation R/R' of one of the rules (2)-(7), such that $R \subseteq G$ and $H = G \cup R'$.*

Then, $G \models_{\rho df_\bot} H$.

Proof. G is satisfiable, hence let $\mathcal{I} = \langle \Delta_{\mathsf{R}}, \Delta_{\mathsf{P}}, \Delta_{\mathsf{C}}, \Delta_L, \mathfrak{P}[\![\cdot]\!], \mathfrak{C}[\![\cdot]\!], \cdot^{\mathcal{I}} \rangle$ be a model of G ($\mathcal{I} \Vdash_{\rho df_\bot} G$) for some assignment A. That is, $\mathcal{I}$ satisfies all the conditions in Definition 2.1. We have to prove that $G \models_{\rho df_\bot} H$, that is, $\mathcal{I} \Vdash_{\rho df_\bot} H$.

We consider only the rules (5)-(7). The theorem has already been proved for groups of rules (1)-(4) in [63, Lemma 31].[4]

[4] In [63, Lemma 31] the authors consider a stronger system in which the predicates sc and sp are reflexive. We drop such properties, hence dropping the corresponding groups (6) and (7) of derivation rules in [63, Table 1]. Hence we need to follow the proof of Lemma 31 in [63] only until the point (4) included.

Rule $(5a)$**.** Let $(c, \bot_{\mathsf{c}}, d) \in R$ for some $R \subseteq G$, $R' = R \cup \{(d, \bot_{\mathsf{c}}, a)\}$, obtained via the application of rule $(5a)$, and $H = G \cup R'$. We have that for every model $\mathcal{I}$ of G, $\mathcal{I} \Vdash_{\rho df_\bot} R$, since $R \subseteq G$. Therefore, $\mathcal{I}$ satisfies $(c, \bot_{\mathsf{c}}, d)$, and, since it is a model of G and is symmetric on $\bot_{\mathsf{c}}$ (see Definition 2.1), we have that $(d^{\mathcal{I}}, c^{\mathcal{I}}) \in \mathfrak{P}[\![{\bot_{\mathsf{c}}}^{\mathcal{I}}]\!]$. That is, $\mathcal{I}$ satisfies $(d, \bot_{\mathsf{c}}, c)$. Hence, from $\mathcal{I} \Vdash_{\rho df_\bot} R'$ and $\mathcal{I} \Vdash_{\rho df_\bot} G$, $\mathcal{I} \Vdash_{\rho df_\bot} H$ follows.

Rule $(5b)$**.** Let $(c, \bot_{\mathsf{c}}, d)$ and (e, sc, c) be in R, for some $R \subseteq G$. Consider $R' = R \cup \{(e, \bot_{\mathsf{c}}, d)\}$, obtained via the application of rule $(5b)$, and $H = G \cup R'$. We have that for every model $\mathcal{I}$ of G, $\mathcal{I} \Vdash_{\rho df_\bot} R$, since $R \subseteq G$. $\mathcal{I}$ satisfies $(c, \bot_{\mathsf{c}}, d)$ and (e, sc, c), and, since it is a model of G, by sc-transitivity of $\mathcal{I}$, $(e^{\mathcal{I}}, d^{\mathcal{I}}) \in \mathfrak{P}[\![{\bot_{\mathsf{p}}}^{\mathcal{I}}]\!]$ follows. That is, $\mathcal{I}$ satisfies $(e, \bot_{\mathsf{c}}, d)$. Hence, since $\mathcal{I} \Vdash_{\rho df_\bot} R'$ and $\mathcal{I} \Vdash_{\rho df_\bot} G$, we have that $\mathcal{I} \Vdash_{\rho df_\bot} H$.

Rule $(5c)$**.** Let $(c, \bot_{\mathsf{c}}, c) \in R$ for some $R \subseteq G$, $R' = R \cup \{(c, \bot_{\mathsf{c}}, d)\}$, obtained via the application of rule $(5c)$, and $H = G \cup R'$. We have that for every model $\mathcal{I}$ of G, $\mathcal{I} \Vdash_{\rho df_\bot} R$, since $R \subseteq G$. Therefore, $\mathcal{I}$ satisfies $(c, \bot_{\mathsf{c}}, c)$, and, since it is a model of G and is c-exhaustive on $\bot_{\mathsf{c}}$ (see Definition 2.1), we have that $(c^{\mathcal{I}}, d^{\mathcal{I}}) \in \mathfrak{P}[\![{\bot_{\mathsf{c}}}^{\mathcal{I}}]\!]$. That is, $\mathcal{I}$ satisfies $(c, \bot_{\mathsf{c}}, d)$. Hence, from $\mathcal{I} \Vdash_{\rho df_\bot} R'$ and $\mathcal{I} \Vdash_{\rho df_\bot} G$, $\mathcal{I} \Vdash_{\rho df_\bot} H$ follows.

Rules $(6a)$, $(6b)$ **and** $(6c)$**.** The argument is analogous to rules $(5a)$, $(5b)$ and $(5c)$

Rule $(7a)$**.** Let (p, dom, c), (q, dom, d), and $(c, \bot_{\mathsf{c}}, d)$ be in R for some $R \subseteq G$, $R' = R \cup \{(p, \bot_{\mathsf{p}}, q)\}$, obtained via the application of rule $(7a)$, and $H = G \cup R'$. We have that for every model $\mathcal{I}$ of G, $\mathcal{I} \Vdash_{\rho df_\bot} R$, since $R \subseteq G$. $\mathcal{I}$ satisfies (p, dom, c), (q, dom, d), and $(c, \bot_{\mathsf{c}}, d)$, that by condition 1 of Disjointness II implies $\mathcal{I} \Vdash_{\rho df_\bot} (p, \bot_{\mathsf{p}}, q)$.

Rule $(7b)$**.** As for rule $(7a)$, just by referring to condition 2 of Disjointness II instead of condition 1 of Disjointness II.

□

The following lemma defines the construction of the *canonical model* for $\rho df_\bot$ graphs. Let $\mathsf{Cl}(G)$ be the closure of G under the application of rules $(2) - (7)$.

Lemma 2.3. *Given a $\rho df_\bot$-graph G, define an* interpretation $\mathcal{I}_G$ *as a tuple*

$$\mathcal{I}_G = \langle \Delta_{\mathsf{R}}, \Delta_{\mathsf{P}}, \Delta_{\mathsf{C}}, \Delta_{\mathsf{L}}, \mathfrak{P}[\![\cdot]\!], \mathfrak{C}[\![\cdot]\!], \cdot^{\mathcal{I}_G} \rangle$$

such that:

1. $\Delta_{\mathsf{R}} := \mathtt{uni}(G) \cup \rho df_\bot$;
2. $\Delta_{\mathsf{P}} := \{p \in \mathtt{uni}(G) \mid (s, p, o) \in \mathsf{Cl}(G)\} \cup \rho df_\bot \cup \{p \in \mathtt{uni}(G) \mid$ *either* (p, sp, q), (q, sp, p), (p, dom, c), (p, range, d), $(p, \bot_{\mathsf{p}}, q)$ *or* $(q, \bot_{\mathsf{p}}, p) \in \mathsf{Cl}(G)\}$;
3. $\Delta_{\mathsf{C}} := \{c \in \mathtt{uni}(G) \mid (x, \mathsf{type}, c) \in \mathsf{Cl}(G)\} \cup \{c \in \mathtt{uni}(G) \mid$ *either* (c, sc, d), (d, sc, c), $(p, \mathtt{dom}, c)$, (p, range, c), $(c, \bot_{\mathsf{c}}, d)$ *or* $(d, \bot_{\mathsf{c}}, c) \in \mathsf{Cl}(G)\}$;
4. $\Delta_{\mathsf{L}} := \mathtt{uni}(G) \cap \mathbf{L}$;
5. $\mathfrak{P}[\![\cdot]\!]$ *is an extension function* $\mathfrak{P}[\![\cdot]\!] \colon \Delta_{\mathsf{P}} \to 2^{\Delta_{\mathsf{R}} \times \Delta_{\mathsf{R}}}$ *s.t.* $\mathfrak{P}[\![p]\!] := \{(s, o) \mid (s, p, o) \in \mathsf{Cl}(G)\}$;
6. $\mathfrak{C}[\![\cdot]\!]$ *is an extension function* $\mathfrak{C}[\![\cdot]\!] \colon \Delta_{\mathsf{C}} \to 2^{\Delta_{\mathsf{R}}}$ *s.t.* $\mathfrak{C}[\![c]\!] := \{x \in \mathtt{uni}(G) \mid (x, \mathsf{type}, c) \in \mathsf{Cl}(G)\}$;
7. $\cdot^{\mathcal{I}_G}$ *is an identity function over* Δ_{R}.

Then, for every $\rho df_\perp$-graph G, $\mathcal{I}_G \Vdash_{\rho df_\perp} G$.

Proof. We need to prove that $\mathcal{I}_G$ satisfies the constraints in Definition 2.1. At first, note that for the conditions **Simple** to **Typing II**, the proof corresponds to the proof of Lemma 32 in [63][5]. So, let us verify the remaining conditions.

Disjointness I:

1. If $(c,d) \in \mathfrak{P}[\![\perp_{\mathsf{c}}{}^{\mathcal{I}_G}]\!]$, then $c, d \in \Delta_{\mathsf{C}}$. This holds by construction of Δ_{C}.
2. $\mathfrak{P}[\![\perp_{\mathsf{c}}{}^{\mathcal{I}_G}]\!]$ is symmetric, *sc*-transitive and *c*-exhaustive over Δ_{C}.
   Symmetry: Let $(c,d) \in \mathfrak{P}[\![\perp_{\mathsf{c}}{}^{\mathcal{I}_G}]\!] = \mathfrak{P}[\![\perp_{\mathsf{c}}]\!]$. By construction of $\mathcal{I}_G$ we have that $(c, \perp_{\mathsf{c}}, d) \in \mathsf{Cl}(G)$, and we also have that $c, d \in \Delta_{\mathsf{C}}$. Due to the closure under rule (5a), $(d, \perp_{\mathsf{c}}, c) \in \mathsf{Cl}(G)$, and by construction of $\mathfrak{P}[\![\cdot]\!]$, $(d,c) \in \mathfrak{P}[\![\perp_{\mathsf{c}}{}^{\mathcal{I}_G}]\!]$.
   sc-*transitivity*: Let $(c,d) \in \mathfrak{P}[\![\perp_{\mathsf{c}}{}^{\mathcal{I}_G}]\!] = \mathfrak{P}[\![\perp_{\mathsf{c}}]\!]$ and $(e,c) \in \mathfrak{P}[\![\mathsf{sc}^{\mathcal{I}_G}]\!] = \mathfrak{P}[\![\mathsf{sc}]\!]$. By construction of $\mathcal{I}_G$ we have that $(c, \perp_{\mathsf{c}}, d), (e, \mathsf{sc}, c) \in \mathsf{Cl}(G)$, and we also have that $c, d, e \in \Delta_{\mathsf{C}}$. Due to the closure under rule (5b), $(e, \perp_{\mathsf{c}}, d) \in \mathsf{Cl}(G)$, and by construction of $\mathfrak{P}[\![\cdot]\!]$, $(e,d) \in \mathfrak{P}[\![\perp_{\mathsf{c}}{}^{\mathcal{I}_G}]\!]$.
   c-*exhaustive*: Let $(c,c) \in \mathfrak{P}[\![\perp_{\mathsf{c}}{}^{\mathcal{I}_G}]\!] = \mathfrak{P}[\![\perp_{\mathsf{c}}]\!]$. By construction of $\mathcal{I}_G$ we have that $(c, \perp_{\mathsf{c}}, c) \in \mathsf{Cl}(G)$ and $c \in \Delta_{\mathsf{C}}$. Due to the closure under rule (5c), $(c, \perp_{\mathsf{c}}, d) \in \mathsf{Cl}(G)$, and by construction of $\mathfrak{P}[\![\cdot]\!]$, $(c,d) \in \mathfrak{P}[\![\perp_{\mathsf{c}}{}^{\mathcal{I}_G}]\!]$ with $d \in \Delta_{\mathsf{C}}$.
3. If $(p,q) \in \mathfrak{P}[\![\perp_{\mathsf{p}}{}^{\mathcal{I}_G}]\!]$, then $p, q \in \Delta_{\mathsf{P}}$. This holds by construction of Δ_{P}.
4. $\mathfrak{P}[\![\perp_{\mathsf{p}}{}^{\mathcal{I}_G}]\!]$ is symmetric, *sp*-transitive and *p*-exhaustive over Δ_{P}. The proof is a rephrasing of the proof of point 2.

Disjointness II:

1. If $(p,c) \in \mathfrak{P}[\![\mathsf{dom}^{\mathcal{I}_G}]\!]$, $(q,d) \in \mathfrak{P}[\![\mathsf{dom}^{\mathcal{I}_G}]\!]$, and $(c,d) \in \mathfrak{P}[\![\perp_{\mathsf{c}}{}^{\mathcal{I}_G}]\!]$, then $(p,q) \in \mathfrak{P}[\![\perp_{\mathsf{p}}{}^{\mathcal{I}_G}]\!]$.
   Indeed, let $(p,c) \in \mathfrak{P}[\![\mathsf{dom}^{\mathcal{I}_G}]\!]$, $(q,d) \in \mathfrak{P}[\![\mathsf{dom}^{\mathcal{I}_G}]\!]$, and $(c,d) \in \mathfrak{P}[\![\perp_{\mathsf{c}}{}^{\mathcal{I}_G}]\!]$. By construction of $\mathfrak{P}[\![\cdot]\!]$, (p, dom, c), (q, dom, d), and $(c, \perp_{\mathsf{c}}, d)$ are in $\mathsf{Cl}(G)$. Moreover, $\mathsf{Cl}(G)$ is closed under rule (7a) and, thus, $(p, \perp_{\mathsf{p}}, q) \in \mathsf{Cl}(G)$, and, by construction of $\mathfrak{P}[\![\cdot]\!]$, $(p,q) \in \mathfrak{P}[\![\perp_{\mathsf{p}}{}^{\mathcal{I}_G}]\!]$.
2. If $(p,c) \in \mathfrak{P}[\![\mathsf{range}^{\mathcal{I}_G}]\!]$, $(q,d) \in \mathfrak{P}[\![\mathsf{range}^{\mathcal{I}_G}]\!]$, and $(c,d) \in \mathfrak{P}[\![\perp_{\mathsf{c}}{}^{\mathcal{I}_G}]\!]$, then $(p,q) \in \mathfrak{P}[\![\perp_{\mathsf{p}}{}^{\mathcal{I}_G}]\!]$.
   The proof is analogous to the previous point: we just need to refer to range and rule (7b) instead of dom and rule (7a), which concludes.

□

Eventually, we have:

Lemma 2.4. *Let G and H be $\rho df_\perp$-graphs. If $G \models_{\rho df_\perp} H$ then $H \subseteq \mathsf{Cl}(G)$.*

Proof. The proof mirrors the proof of Lemma 33 in [63]. In particular, consider the interpretation

$$\mathcal{I}_G = \langle \Delta_{\mathsf{R}}, \Delta_{\mathsf{P}}, \Delta_{\mathsf{C}}, \Delta_{\mathsf{L}}, \mathfrak{P}[\![\cdot]\!], \mathfrak{C}[\![\cdot]\!], \cdot^{\mathcal{I}_G} \rangle$$

as defined in Lemma 2.3. Therefore, as both $\mathcal{I}_G \Vdash_{\rho df_\perp} G$ and $G \models_{\rho df_\perp} H$ hold, we have $\mathcal{I}_G \Vdash_{\rho df_\perp} H$ by Definition 2.2. Therefore, for each $(s,p,o) \in H$, $p^{\mathcal{I}_G} \in \Delta_{\mathsf{P}}$ and $(s^{\mathcal{I}_G}, o^{\mathcal{I}_G}) \in \mathfrak{P}[\![p^{\mathcal{I}_G}]\!]$. Moreover, by construction $p^{\mathcal{I}_G} = p$, and $\mathfrak{P}[\![p^{\mathcal{I}_G}]\!] = \mathfrak{P}[\![p]\!] = \{(s,o) \mid (s,p,o) \in \mathsf{Cl}(G)\}$. Finally, since $(s^{\mathcal{I}_G}, o^{\mathcal{I}_G}) \in \mathfrak{P}[\![p^{\mathcal{I}_G}]\!]$, we have that $(s^{\mathcal{I}_G}, p^{\mathcal{I}_G}, o^{\mathcal{I}_G})) \in \mathsf{Cl}(G)$, *i.e.*, $(s,p,o) \in \mathsf{Cl}(G)$, for each $(s,p,o) \in H$. Therefore, $H \subseteq \mathsf{Cl}(G)$, which concludes. □

[5] With the minor difference that in [63, Lemma32] the authors impose also reflexivity to the interpretations of the predicates sc and sp and consider the associated derivation rules, while here we do not.

Eventually, we can prove the main theorem of this section.

Proof of Theorem 2.1. The proof mirrors the proof of Theorem 8 in [63]. From Lemma 2.4, $G \models_{\rho df_\perp} H$ implies that H can be obtained from $\mathsf{Cl}(G)$ using rule (1). Thus, since $G \vdash_{\rho df_\perp} \mathsf{Cl}(G)$, it follows that $G \vdash_{\rho df_\perp} H$. Therefore Theorem 2.1 follows from Lemmas 2.2 and 2.4. □

Please note that, like in classical ρdf, $\rho df_\perp$-graphs are always satisfiable.

Corollary 2.5. *A $\rho df_\perp$-graph G is always satisfiable.*

Proof. This is an immediate consequence of Lemma 2.3. □

2.4 Some interesting derived inference rules

In what follows, we illustrate the derivation of some interesting rules of inference. To start with, note that the triples $(d, \perp_\mathsf{c}, d)$ and $(q, \perp_\mathsf{p}, q)$ are particularly significant: indeed, the intended meaning of *e.g.*, $(d, \perp_\mathsf{c}, d)$ is that 'concept/class d is empty'.

Some derived inference rules that will turn out to be useful are the following:

Empty Subclass:

$$\frac{(A, \mathsf{sc}, B) \qquad (A, \mathsf{sc}, C) \qquad (B, \perp_\mathsf{c}, C)}{(A, \perp_\mathsf{c}, A)} \text{ (EmptySC)}$$

Here is the derivation:

$$\dfrac{\text{(5a)}\ \dfrac{\text{(5b)}\ \dfrac{(A, \mathsf{sc}, B) \qquad (B, \perp_\mathsf{c}, C)}{(A, \perp_\mathsf{c}, C)}}{(C, \perp_\mathsf{c}, A)} \qquad (A, \mathsf{sc}, C)}{(A, \perp_\mathsf{c}, A)} \text{ (5b)}$$

A special case of rule (EmptySC) is obtained by imposing $B = C$, shows that if a class is empty, also all its subclasses are empty.

$$\frac{(A, \mathsf{sc}, B) \qquad (B, \perp_\mathsf{c}, B)}{(A, \perp_\mathsf{c}, A)} \text{ (EmptySC')}$$

Empty Subpredicate:

$$\frac{(A, \mathsf{sp}, B) \qquad (A, \mathsf{sp}, C) \qquad (B, \perp_\mathsf{p}, C)}{(A, \perp_\mathsf{p}, A)} \text{ (EmptySP)}$$

Here is the derivation:

$$\dfrac{\text{(6a)}\ \dfrac{\text{(6b)}\ \dfrac{(A, \mathsf{sp}, B) \qquad (B, \perp_\mathsf{p}, C)}{(A, \perp_\mathsf{p}, C)}}{(C, \perp_\mathsf{p}, A)} \qquad (A, \mathsf{sp}, C)}{(A, \perp_\mathsf{p}, A)} \text{ (6b)}$$

As for (EmptySC), by imposing $B = C$ we obtain a special case of the rule (EmptySP) showing that if a property is empty, so are all its subproperties.

$$\frac{(A, \mathsf{sc}, B) \qquad (B, \bot_{\mathsf{p}}, B)}{(A, \bot_{\mathsf{p}}, A)} \text{ (EmptySP')}$$

Conflicting Domain:

$$\frac{(A, \mathsf{dom}, C) \qquad (A, \mathsf{dom}, X) \qquad (C, \bot_{\mathsf{c}}, X)}{(A, \bot_{\mathsf{p}}, A)} \text{ (7a')}$$

This is a special case of the rule (7a) in which $B = A$.

Conflicting Range:

$$\frac{(A, \mathsf{range}, C) \qquad (A, \mathsf{range}, X) \qquad (C, \bot_{\mathsf{c}}, X)}{(A, \bot_{\mathsf{p}}, A)} \text{ (7b')}$$

Similarly as above, this is a special case of the rule (7b) in which $B = A$.

3 Defeasible $\rho df_\bot$

We now introduce the possibility of modelling defeasible information in the RDFS framework. A framework without any notion of potential conflict between different pieces of information would have not allowed to model *presumptive reasoning*, that is, the kind of uncertain reasoning in which we proceed assuming that what we consider as *typically true* holds whenever it is not in conflict with the other information at our disposal.

3.1 Syntax

We will consider defeasibility only w.r.t. the predicates sc and sp only and introduce the notion of *defeasible triple* defined next.

Definition 3.1 (Defeasible triple)**.** *A* defeasible triple *is of the form*

$$\delta = \langle s, p, o \rangle \in \mathbf{UL} \times \{\mathsf{sc}, \mathsf{sp}\} \times \mathbf{UL} \ ,$$

where $s, o \notin \rho df_\bot$*.*

The intended meaning of *e.g.*, $\langle c, \mathsf{sc}, d \rangle$ is "Typically, an instance of c is also an instance of b". For example, if b is interpreted as the class of birds and f as the class of flying creatures, $\langle b, \mathsf{sc}, f \rangle$ represents the defeasible statement "Typically, birds fly". Analogously, $\langle p, \mathsf{sp}, q \rangle$ is read as "Typically, a pair related by p is also related by q". For example, if p represents the predicate *being the parent of* and b the predicate *being blood-related to*, the triple $\langle p, \mathsf{sp}, b \rangle$ represents the defeasible statement "Generally, the parents are blood-related to their children".

In the following, we use the notation $[\![\cdot, \cdot, \cdot]\!]$ to indicate either $(\cdot, \cdot, \cdot)$ or $\langle \cdot, \cdot, \cdot \rangle$ (that is, $[\![\cdot, \cdot, \cdot]\!] \in \{(\cdot, \cdot, \cdot), \langle \cdot, \cdot, \cdot \rangle\}$).

Remark 3.1. *Note that in practice a triple* $\langle c, \mathsf{sc}, d\rangle$ *may be represented as* (c, sc_t, d), *where* sc_t *is a new symbol indicating defeasible class inclusion. Similarly,* $\langle p, \mathsf{sp}, q\rangle$ *may be represented as* (p, sp_t, q), *where* sp_t *is a new symbol indicating defeasible property inclusion. Therefore, both defeasible triples could have been represented in* $\rho df_\perp \cup \{\mathsf{sc}_t, \mathsf{sp}_t\}$. *While certainly this is an option for a practical implementation, for ease of presentation, we prefer to stick to the former notation.*

A *defeasible graph* is a set $G = G^{str} \cup G^{def}$, where G^{str} is a $\rho df_\perp$-graph and G^{def} is a set of defeasible triples. Given two defeasible graphs G and G', G is a sub-graph of G' iff $G \subseteq G'$.

Given a defeasible graph $G = G^{str} \cup G^{def}$, its *strict counterpart* is the graph

$$G^s := G^{str} \cup \{(s, p, o) \mid \langle s, p, o\rangle \in G^{def}\} \ . \tag{1}$$

Generally speaking, given a defeasible graph G, we need to define how to reason with it. As mentioned above, in presumptive reasoning it is considered rational to reason classically with defeasible information in case no conflicts arise; on the other hand, if we have to deal with conflictual information we need to refer to some form of defeasible reasoning to resolve such conflicts.

To do so, first of all, we need to define the notion of *conflict* in our framework.

Definition 3.2 (conflict)**.** *Let G be a defeasible graph. G has a* conflict *if, for some term* t, *either* $G^s \vdash_{\rho df_\perp} (t, \perp_{\mathsf{c}}, t)$ *or* $G^s \vdash_{\rho df_\perp} (t, \perp_{\mathsf{p}}, t)$ *holds.*

Informally, we consider that there is a conflict in a defeasible graph if, treating the defeasible triples as classical triples, we have that some term t may be interpreted as being 'empty'.[6] To give a sense of the rationale behind this definition, let's see how it behaves w.r.t. the typical *penguin example* from the non-monotonic reasoning literature.

Example 3.1 (Penguin example)**.** *Consider the following graph:*

$$G = \{(p, \mathsf{sc}, b), \langle b, \mathsf{sc}, f\rangle, (p, \mathsf{sc}, e), (e, \perp_{\mathsf{c}}, f)\} \ ,$$

where p, b, f, e *stand for* penguins, birds, flying creatures, non-flying creatures*, respectively. As it is well-known, the penguin example shows that exceptional subclasses cannot be appropriately be modelled with classical reasoning, since, in the specific case, we would be forced to conclude that penguins can fly and cannot fly at the same time, that is, penguins cannot exist. In our context, we have the following. The strict counterpart of G is*

$$G^s = \{(p, \mathsf{sc}, b), (b, \mathsf{sc}, f), (p, \mathsf{sc}, e), (e, \perp_{\mathsf{c}}, f)\}$$

and from G^s *we can indeed derive:*

$$\dfrac{\text{(3a)}\ \dfrac{(p, \mathsf{sc}, b) \qquad (b, \mathsf{sc}, f)}{(p, \mathsf{sc}, f)} \qquad (p, \mathsf{sc}, e) \qquad (e, \perp_{\mathsf{c}}, f)}{(p, \perp_{\mathsf{c}}, p)}\ \text{(EmptySC)}$$

That is, reasoning with a strict $\rho df_\perp$*-graph penguins may be interpreted as 'empty'.*

[6]Note that the concept of conflict that we define here is related to the notion of *incoherence* in OWL formalism [75]: an OWL ontology is incoherent if a concept introduced in the vocabulary turns out to be empty.

3.2 Semantics

An interpretation for a defeasible graphs G is composed by a set of $\rho df_\bot$ interpretations, ranked accordingly to how much they conform to our expectations.

Definition 3.3 (Ranked $\rho df_\bot$ Interpretations)**.** *A ranked interpretation is a pair $\mathcal{R} = (\mathscr{M}, r)$, where $\mathscr{M}$ is the set of all $\rho df_\bot$ interpretations defined on a fixed set of domains $\Delta_R, \Delta_P, \Delta_C, \Delta_L$, and r is a ranking functions over $\mathscr{M}$* [7]

$$r : \mathscr{M} \mapsto \mathbb{N} \cup \{\infty\}$$

satisfying a convexity property:

- *there is an interpretation $\mathcal{I} \in \mathscr{M}$ s.t. $r(\mathcal{I}) = 0$;*
- *for each $i > 0$, if there is an interpretation $\mathcal{I} \in \mathscr{M}$ s.t. $r(\mathcal{I}) = i$, then there is an interpretation $\mathcal{I}' \in \mathscr{M}$ s.t. $r(\mathcal{I}') = (i-1)$.*

Informally, the intuition behind ranking interpretations is that, given two interpretations $\mathcal{I}, \mathcal{I}' \in \mathscr{M}$, $r(\mathcal{I}) < r(\mathcal{I}')$ indicates that the interpretation $\mathcal{I}$ is more in line with our expectations than the interpretation $\mathcal{I}'$.

Now, given $\mathcal{R} = (\mathscr{M}, r)$, let $\mathscr{M}_\mathbb{N}$ be the set of elements in $\mathscr{M}$ with rank lower that ∞, that is,

$$\mathscr{M}_\mathbb{N} = \{\mathcal{I} \in \mathscr{M} \mid r(\mathcal{I}) \in \mathbb{N}\} .$$

Informally, in $\mathcal{R}$ the $\rho df_\bot$-interpretations with rank infinite are simply considered impossible, and the satisfaction relation, which we will define next is determined referring only to the $\rho df_\bot$-interpretations in $\mathscr{M}_\mathbb{N}$. Given a ranked interpretation $\mathcal{R} = (\mathscr{M}, r)$ and a term t, let $\mathsf{c\_}\min(t, \mathcal{R})$ be the set of the most expected interpretations in $\mathscr{M}$ in which t (interpreted as class) is not empty, that is,

$$\begin{aligned}\mathsf{c\_}\min(t,\mathcal{R}) \quad = \quad & \{\mathcal{I} \in \mathscr{M}_\mathbb{N} \mid \mathcal{I} \not\Vdash_{\rho df_\bot} (t, \bot_\mathsf{c}, t) \text{ and there is no } \mathcal{I}' \in \mathscr{M}_\mathbb{N} \text{ s.t.} \\ & \mathcal{I}' \not\Vdash_{\rho df_\bot} (t, \bot_\mathsf{c}, t) \text{ and } r(\mathcal{I}') < r(\mathcal{I})\} .\end{aligned}$$

Analogously, for a term t interpreted as predicate, we define

$$\begin{aligned}\mathsf{p\_}\min(t,\mathcal{R}) \quad = \quad & \{\mathcal{I} \in \mathscr{M}_\mathbb{N} \mid \mathcal{I} \not\Vdash_{\rho df_\bot} (t, \bot_\mathsf{p}, t) \text{ and there is no } \mathcal{I}' \in \mathscr{M}_\mathbb{N} \text{ s.t.} \\ & \mathcal{I}' \not\Vdash_{\rho df_\bot} (t, \bot_\mathsf{p}, t) \text{ and } r(\mathcal{I}') < r(\mathcal{I})\} .\end{aligned}$$

Definition 3.4 (Ranked satisfaction)**.** *For every triple (s, p, o), a ranked interpretation $\mathcal{R} = (\mathscr{M}, r)$* satisfies *$(s, p, o)$ if (s, p, o) is satisfied by every $\rho df_\bot$-interpretation in $\mathscr{M}$, that is,*

$$\mathcal{R} \Vdash_{\rho df_\bot} (s,p,o) \textit{ iff } \mathcal{I} \Vdash_{\rho df_\bot} (s,p,o) \textit{ for every } \mathcal{I} \in \mathscr{M}_\mathbb{N} .$$

For every defeasible triple of the form $\langle s, p, o\rangle$, the notion of a ranked interpretation $\mathcal{R} = (\mathscr{M}, r)$ satisfying *$\langle s, p, o\rangle$, denoted $\mathcal{R} \Vdash_{\rho df_\bot} \langle s, p, o\rangle$, is defined as follows:*

$$\begin{aligned}\mathcal{R} \Vdash_{\rho df_\bot} \langle c, \mathsf{sc}, s\rangle \quad & \textit{iff} \quad \mathcal{I} \Vdash_{\rho df_\bot} (c, \mathsf{sc}, d) \textit{ for every } \mathcal{I} \in \mathsf{c\_}\min(c, \mathcal{R}) \\ \mathcal{R} \Vdash_{\rho df_\bot} \langle p, \mathsf{sp}, q\rangle \quad & \textit{iff} \quad \mathcal{I} \Vdash_{\rho df_\bot} (p, \mathsf{sp}, q) \textit{ for every } \mathcal{I} \in \mathsf{p\_}\min(p, \mathcal{R}) .\end{aligned}$$

Given a defeasible $\rho df_\bot$-graph $G = G^{str} \cup G^{def}$, a ranked interpretation $\mathcal{R} = (\mathscr{M}, r)$ is a model *of G (denoted $\mathcal{R} \Vdash_{\rho df_\bot} G$) if $\mathcal{R} \Vdash_{\rho df_\bot} (s, p, o)$ for every $(s, p, o) \in G^{str}$, and $\mathcal{R} \Vdash_{\rho df_\bot} \langle s, p, o\rangle$ for every $\langle s, p, o\rangle \in G^{def}$.*

[7] We will assume that $0 \in \mathbb{N}$.

The intuition behind this definition is that the triple $\langle c, \mathsf{sc}, s\rangle$ holds in a ranked interpretation if (c, sc, s) holds in the most expected $\rho df_\perp$-interpretations in which c is not an empty class. The intuition follows the same line of the original propositional construction [58], and its DL reformulations [25, 22, 42]. As in the propositional and DL constructions, once we have defined the notion of ranked interpretation to model defeasible information, the problem is to decide which kind of entailment relation, that is, what kind of defeasible reasoning, we would like to model. Despite it is recognised that there are multiple available options according to the kind of properties we want to satisfy [56, 58, 57, 67, 26, 23, 42], it is generally recognised that Lehmann and Magidor's RC [58] is the fundamental construction in the area, and most of the other proposed systems can be built as refinements of it. We recall that RC models the so-called *Presumption of Typicality* [59, p.4], that is the reasoning principle imposing that, if we are not informed about any exceptional property, we presume that we are dealing with a typical situation. The essential behaviour characterising the presumption of typicality is that a subclass that does not show any exceptional property inherits all the typical properties of the superclass. From a semantics point of view the definition of RC can be obtained via various equivalent definitions [52, 58]: here we opt for the characterisation of RC using the minimal ranked model [21, 67] and, in particular, we consider the characterisation given by Giordano et al. [42], which we believe appropriate for our defeasible RDF framework.

In the following, given a defeasible graph $G = G^{str} \cup G^{def}$ and its strict counterpart G^s (see Eq. 1), let the interpretation $\mathcal{I}_{G^s}$

$$\mathcal{I}_{G^s} := \langle \Delta_\mathsf{R}, \Delta_\mathsf{P}, \Delta_\mathsf{C}, \Delta_\mathsf{L}, \mathfrak{P}[\![\cdot]\!], \mathfrak{C}[\![\cdot]\!], \cdot^{\mathcal{I}_{G^s}}\rangle \tag{2}$$

be defined from G^s as in Lemma 2.3, and consider the domains $\Delta_\mathsf{R}, \Delta_\mathsf{P}, \Delta_\mathsf{C}, \Delta_\mathsf{L}$ in it. With $\mathscr{M}^G$ we denote the set of all $\rho df_\perp$-interpretations defined over such domains, and we indicate a ranked interpretation build over $\mathscr{M}^G$ as $\mathcal{R}^G = (\mathscr{M}^G, r)$. In the following, if clear from the context that a ranked interpretation is built from a graph G as here indicated, we may omit the superscript G in $\mathcal{R}^G = (\mathscr{M}^G, r)$. [8]

Now, given a defeasible graph G, let $\mathfrak{I}_G$ be the set of the ranked interpretations $\mathcal{R}^G$ and let $\mathfrak{R}_G$ be the elements of $\mathfrak{I}_G$ that are also models for G, that is,

$$\mathfrak{R}_G := \{\mathcal{R} \in \mathfrak{I}_G \mid \mathcal{R} \Vdash_{\rho df_\perp} G\} \ . \tag{3}$$

Note that by Lemma 2.3, $\mathfrak{R}_G$ is not empty. Moreover, since the domains in Equation 2 are finite, $\mathscr{M}^G$ is finite as well. Additionally, as a ranked interpretation $\mathcal{R}^G = (\mathscr{M}^G, r)$ build over $\mathscr{M}^G$ has to satisfy the convexity property (see Definition 3.3), the ranking function r is bounded by $|\mathscr{M}^G|$.[9] Therefore, $\mathfrak{I}_G$ is a finite and, thus, so is $\mathfrak{R}_G$.

Please note that by construction $\mathcal{R}, \mathcal{R}' \in \mathfrak{R}_G$ differ only w.r.t. the involved ranking functions r, r', respectively, which induces the following order over the ranked models in $\mathfrak{R}_G$.

Definition 3.5 (Presumption ordering $\preceq$)**.** *Let $\mathcal{R} = (\mathscr{M}, r)$, $\mathcal{R}' = (\mathscr{M}, r')$, and $\mathcal{R}, \mathcal{R}' \in \mathfrak{R}_G$. We define*

1. *$\mathcal{R} \preceq \mathcal{R}'$ iff for every $\mathcal{I} \in \mathscr{M}$, $r(\mathcal{I}) \leqslant r'(\mathcal{I})$.*
2. *$\mathcal{R} \prec \mathcal{R}'$ iff $\mathcal{R} \preceq \mathcal{R}'$ and $\mathcal{R}' \not\preceq \mathcal{R}$.*
3. *$\min_\preceq(\mathfrak{R}_G) := \{\mathcal{R} \in \mathfrak{R}_G \mid$ there is no $\mathcal{R}' \in \mathfrak{R}_G$ s.t. $\mathcal{R}' \prec R\}$.*

The set $\min_\preceq(\mathfrak{R}_G)$ contains the ranked models of G in which the $\rho df_\perp$-interpretations are "pushed down" as much as possible in the ranking, that is, they are considered as typical as possible.

We next show that actually there is an unique minimal ranked model for a defeasible graph.

Proposition 3.1. *For every defeasible graph G, $|\min_\preceq(\mathfrak{R}_G)| = 1$.*

[8] Keep in mind that the domains of all interpretations occurring in $\mathcal{R}^G$ are defined as in Equation 2.

[9] That is, the maximal rank of any interpretation in $\mathscr{M}^G$ cannot exceed $|\mathscr{M}^G|$.

Proof. Assume to the contrary that $|\min_{\preceq}(\mathfrak{R}_G)| > 1$, *i.e.*, there are $\mathcal{R} = (\mathscr{M}, r)$, $\mathcal{R}' = (\mathscr{M}, r')$, with $\mathcal{R}, \mathcal{R}' \in \min_{\preceq}(\mathfrak{R}_G)$ and $\mathcal{R} \neq \mathcal{R}'$.

It cannot be the case that $\mathcal{R} \preceq \mathcal{R}'$ and $\mathcal{R}' \preceq \mathcal{R}$, since that would imply that $\mathcal{R} = \mathcal{R}'$. Hence it must be the case that $\mathcal{R}$ and $\mathcal{R}'$ are incomparable w.r.t. $\preceq$; that is, there are at least two interpretations $\mathcal{I}, \mathcal{J}$ in $\mathscr{M}$ s.t. $r(\mathcal{I}) < r'(\mathcal{I})$ and $r'(\mathcal{J}) < r(\mathcal{J})$.

Now, consider $\mathcal{R}^\star = (\mathscr{M}, r^\star)$, with $r^\star(\mathcal{I}) = \min\{r(\mathcal{I}), r'(\mathcal{I})\}$ for every $\mathcal{I} \in \mathscr{M}$. Clearly, $\mathcal{R}^\star \prec \mathcal{R}$ and $\mathcal{R}^\star \prec \mathcal{R}'$. As next, let us prove that that $\mathcal{R}^\star \Vdash_{\rho df_\perp} G$. At first, note that $\mathcal{R}^\star \Vdash_{\rho df_\perp} G^{str}$ holds as the satisfaction of G^{str} depends only on $\mathscr{M}$ and not on the ranks. At second, assume to the contrary that $\mathcal{R}^\star \nVdash_{\rho df_\perp} G^{def}$, that is, there is a defeasible triple $\langle s, p, o\rangle \in G$ such that $\mathcal{R}^\star \nVdash_{\rho df_\perp} \langle s, p, o\rangle$ and let's assume that $\langle s, p, o\rangle$ is of the form $\langle c, \mathsf{sc}, d\rangle$ (the proof for the case $\langle p, \mathsf{sp}, q\rangle$ is similar). That means that there is $\bar{\mathcal{I}} \in \mathsf{c}\_\min(c, \mathcal{R}^\star)$ s.t. $\bar{\mathcal{I}} \nVdash_{\rho df_\perp} (c, \mathsf{sc}, d)$. But, by the definition of $r^\star$, either $\bar{\mathcal{I}} \in \mathsf{c}\_\min(c, \mathcal{R})$ or $\bar{\mathcal{I}} \in \mathsf{c}\_\min(c, \mathcal{R}')$ and, thus, either $\mathcal{R} \nVdash_{\rho df_\perp} \langle c, \mathsf{sc}, d\rangle$ or $\mathcal{R}' \nVdash_{\rho df_\perp} \langle c, \mathsf{sc}, d\rangle$ holds, against the hypothesis that both $\mathcal{R}$ and $\mathcal{R}'$ are ranked models of G. As a consequence, $\mathcal{R}^\star \Vdash_{\rho df_\perp} G^{def}$ has to hold and, thus, $\mathcal{R}^\star \Vdash_{\rho df_\perp} G$. That is, $\mathcal{R}^\star \in \mathfrak{R}_G$. But then, it can not be the case that $\mathcal{R}, \mathcal{R}' \in \min_{\preceq}(\mathfrak{R}_G)$ as there is $\mathcal{R}^\star \in \mathfrak{R}_G$ with $\mathcal{R}^\star \prec \mathcal{R}$ and $\mathcal{R}^\star \prec \mathcal{R}'$, which is against our hypothesis. Therefore, there can not be two distinct ranked models $\mathcal{R}, \mathcal{R}'$ in $\min_{\preceq}(\mathfrak{R}_G)$ and, as $\mathfrak{R}_G$ is not empty, $|\min_{\preceq}(\mathfrak{R}_G)| = 1$ has to hold. □

In the following, the unique $\preceq$-minimal ranked model in $\mathfrak{R}_G$ is called the *minimal G-model* and is denoted with $\mathcal{R}_{\min G}$, *i.e.*, $\min_{\preceq}(\mathfrak{R}_G) = \{\mathcal{R}_{\min G}\}$.

Definition 3.6 (Minimal Entailment). *Given a defeasible graph G and the corresponding minimal G-model $\mathcal{R}_{\min G}$ of G. A defeasible graph G* minimally entails *a triple $[\![s, p, o]\!]$, denoted $G \models_{\min} [\![s, p, o]\!]$, iff $\mathcal{R}_{\min G} \Vdash_{\rho df_\perp} [\![s, p, o]\!]$.*

Using the minimal ranking in $\mathcal{R}_{\min G}$ we can also define the *height* of a term, indicating at which level of exceptionality a term t is not disjoint to itself, respectively as a class or as a predicate. This corresponds to the minimal rank, that is, the rank in $\mathcal{R}_{\min G}$, in which we encounter a $\rho df_\perp$-interpretation that does not satisfy $(t, \perp_{\mathsf{c}}, t)$, or, respectively, $(t, \perp_{\mathsf{p}}, t)$.

Definition 3.7 (Height). *Let $G = G^{str} \cup G^{def}$ be a defeasible graph, with $\mathcal{R}_{\min G} = \{\mathscr{M}, r\}$ being its minimal model, and let t be a term in G. The* c-height *of t corresponds to the lowest rank $r(\mathcal{I})$ of some $\mathcal{I} \in \mathscr{M}$ s.t. $\mathcal{I} \nVdash_{\rho df_\perp} (t, \perp_{\mathsf{c}}, t)$, that is,*

$$h^{\mathsf{c}}_G(t) = \begin{cases} \infty, & \text{if } \mathcal{I} \Vdash_{\rho df_\perp} (t, \perp_{\mathsf{c}}, t) \text{ for every } \mathcal{I} \in \mathscr{M}_{\mathbb{N}} \\ \min\{r(\mathcal{I}) \mid \mathcal{I} \in \mathscr{M} \text{ and } \mathcal{I} \nVdash_{\rho df_\perp} (t, \perp_{\mathsf{c}}, t)\}, & \text{otherwise.} \end{cases}$$

Analogously, the p-height *of t corresponds to the lowest rank $r(\mathcal{I})$ of some $\mathcal{I} \in \mathscr{M}$ s.t. $\mathcal{I} \nVdash_{\rho df_\perp} (t, \perp_{\mathsf{p}}, t)$*

$$h^{\mathsf{p}}_G(t) = \begin{cases} \infty, & \text{if } \mathcal{I} \Vdash_{\rho df_\perp} (t, \perp_{\mathsf{p}}, t) \text{ for every } \mathcal{I} \in \mathscr{M}_{\mathbb{N}} \\ \min\{r(\mathcal{I}) \mid \mathcal{I} \in \mathscr{M} \text{ and } \mathcal{I} \nVdash_{\rho df_\perp} (t, \perp_{\mathsf{p}}, t)\}, & \text{otherwise.} \end{cases} \quad H$$

Also, we define the height *$h(\mathcal{R})$ of a ranked interpretation $\mathcal{R}$ as the highest finite rank of the $\rho df_\perp$-interpretations in it. That is, let $\mathcal{R} = (\mathscr{M}, r)$ be a ranked model, then*

$$h(\mathcal{R}) := \max\{r(\mathcal{I}) \mid \mathcal{I} \in \mathscr{M}_{\mathbb{N}}\}\ .$$

Example 3.2. *Consider the graph F, extending the graph G from Example 3.1 with the triple (r, sc, b).*

$$F = \{(p, \mathsf{sc}, b), (r, \mathsf{sc}, b), \langle b, \mathsf{sc}, f\rangle, (p, \mathsf{sc}, e), (e, \perp_{\mathsf{c}}, f)\}\ ,$$

where r stands for robins. *Its strict counterpart is*

$$F^s = \{(p, \mathsf{sc}, b), (r, \mathsf{sc}, b), (b, \mathsf{sc}, f), (p, \mathsf{sc}, e), (e, \perp_{\mathsf{c}}, f)\}\ .$$

Given F^s, we define the domains used for the models in $\mathfrak{R}_F$ following the construction in Lemma 2.3. The minimal model $\mathcal{R}_{\min F} \in \mathfrak{R}_F$ will be a model of height 1: all the $\rho df_\bot$-interpretations that satisfy F^s will have height 0; all the $\rho df_\bot$-interpretations that satisfy F^{str} but not (b, sc, f) will have height 1, and all the $\rho df_\bot$-interpretations that do not satisfy F^{str} will have infinite height.

As in Example 3.1, F^s implies $(p, \bot_\mathsf{c}, p)$. Also, it is easy to check that F^s does not imply $(b, \bot_\mathsf{c}, b)$ and does not imply $(r, \bot_\mathsf{c}, r)$, since we are informed that penguins do not fly, but we are not informed that robins do not fly (there is no (r, sc, e) in our graph); also, F^{str} does not imply $(p, \bot_\mathsf{c}, p)$, since F^{str} does not contain (b, sc, f) anymore. The resulting configuration of $\mathcal{R}_{\min F}$ is such that: all the $\rho df_\bot$-interpretations with height 0 satisfy $(p, \bot_\mathsf{c}, p)$, but not all of them satisfy $(b, \bot_\mathsf{c}, b)$ or $(r, \bot_\mathsf{c}, r)$; not all the $\rho df_\bot$-interpretations with height 1 satisfy $(p, \bot_\mathsf{c}, p)$.

It is easy to check that $\mathcal{R}_{\min F}$ is a model of F: all the $\rho df_\bot$-interpretations with finite height satisfy the strict part F^{str}; The minimal interpretations that do not satisfy $(b, \bot_\mathsf{c}, b)$ have height 0, and consequently satisfy (b, sc, f), hence the defeasible triple (b, sc, f) is satisfied too by $\mathcal{R}_{\min F}$.

Note that there cannot be a model of F that is preferred to $\mathcal{R}_{\min F}$: if we move any $\rho df_\bot$-interpretation from height 1 to height 0, the resulting model would not satisfy $\langle b, \mathsf{sc}, f\rangle$ anymore, and if we move any $\rho df_\bot$-interpretation from height ∞ to any finite height, the resulting model would not satisfy F^{str}.

Being a minimal model of F, $\mathcal{R}_{\min F}$ satisfies the presumption of typicality. To check this, it suffices to determine what we can derive about robins: since robins do not have any exceptional property, they should inherit all the typical properties of birds, and we should be able to derive that they presumably fly. We have seen that at rank 0 all interpretations satisfy (r, sc, b) and (b, sc, f), but there are some interpretations that do not satisfy $(r, \bot_\mathsf{c}, r)$. Consequently, according to Definition 3.4, $\mathcal{R}_{\min F} \Vdash_{\rho df_\bot} \langle r, \mathsf{sc}, f\rangle$, that is, $F \models_{\min} \langle r, \mathsf{sc}, f\rangle$, as desired.

3.3 Exceptionality

Minimal entailment defines the semantics. We next define a decision procedure for it. To do so, we define the notion of *exceptionality*, a reformulation in our context of a property that is fundamental for RC [58]. Informally, a class t (or, respectively, a predicate t) is exceptional w.r.t. a defeasible graph if there is no typical situation in which t can be populated with some instance. Formally it corresponds to saying that in every ranked model of the graph, all the $\rho df_\bot$-interpretations with height 0 satisfy $(t, \bot_\mathsf{c}, t)$ (respectively, $(t, \bot_\mathsf{p}, t)$).

Definition 3.8 (Exceptionality)**.** *Let G be a defeasible $\rho df_\bot$-graph, $\mathcal{R} = (\mathscr{M}, r)$ be a ranked model in $\mathfrak{R}_G$ and t be a term.*

1. *We say that t is* c-exceptional *(resp.* p-exceptional*) w.r.t. $\mathcal{R}$ if for every $\mathcal{I} \in \mathscr{M}$ s.t. $r(\mathcal{I}) = 0$, we have that $\mathcal{I} \Vdash_{\rho df_\bot} (t, \bot_\mathsf{c}, t)$ (resp. $\mathcal{I} \Vdash_{\rho df_\bot} (t, \bot_\mathsf{p}, t)$).*

2. *We say that t is* c-exceptional *(resp.* p-exceptional*) w.r.t. G if it is c-exceptional (resp.* p-exceptional*) w.r.t. all $\mathcal{R} \in \mathfrak{R}_G$.*

It turns out that in order to check exceptionality w.r.t. a graph it is sufficient to refer to the minimal model of the graph.

Proposition 3.2. *A term t is c-exceptional (resp. p-exceptional) w.r.t. a defeasible graph G iff it is c-exceptional (resp. p-exceptional) w.r.t. $\mathcal{R}_{\min G}$.*

Proof. Let t be c-exceptional.

$\Rightarrow$) Immediate from the definition of exceptionality.

$\Leftarrow$) We proceed by contradiction: let t be c-exceptional w.r.t. $\mathcal{R}_{\min G}$, and assume that there is a model $\mathcal{R} = (\mathscr{M}, r_{\mathcal{R}}) \in \mathfrak{R}_G$ s.t. t is not c-exceptional w.r.t. $\mathcal{R}$. That is, there is an $\mathcal{I} \in \mathscr{M}$ s.t. $r_{\mathcal{R}}(\mathcal{I}) = 0$ and $\mathcal{I} \not\Vdash_{\rho df_\bot} (t, \bot_{\mathsf{c}}, t)$. In such a case, by Definition 3.5 and Proposition 3.1, we would have that $\mathcal{I}$ has rank 0 also in $\mathcal{R}_{\min G}$. Consequently t cannot be c-exceptional w.r.t. $\mathcal{R}_{\min G}$, which contradicts our assumption.

The proof is analogous if t is p-exceptional. □

The following proposition represents the bridge between the semantic notion of exceptionality and $\rho df_\bot$ decidability, and it will be central in what follows.

Proposition 3.3. *A term t is c-exceptional (resp., p-exceptional) w.r.t. a defeasible graph G iff $G^s \vdash_{\rho df_\bot} (t, \bot_{\mathsf{c}}, t)$ (resp., $G^s \vdash_{\rho df_\bot} (t, \bot_{\mathsf{p}}, t)$).*

In order to prove the above proposition, we introduce the notion of proof tree and prove some lemmas beforehand. To start with, we reformulate the classical notion of proof tree for $\rho df_\bot$.

Definition 3.9 ($\rho df_\bot$ proof tree)**.** *A $\rho df_\bot$ proof tree is a finite tree in which*

- *each node is a $\rho df_\bot$ triple;*
- *every node is connected to the node(s) immediately above through one of the inference rules (1)-(7) presented in Section 2.3.*

Let T be a $\rho df_\bot$ proof tree, H be the set of the triples appearing as top nodes (called leaves*) and let (s, p, o) be the triple appearing in the unique bottom node (called* root*). Then T is a $\rho df_\bot$ proof tree from H to (s, p, o).*

For istance, in Example 3.1 we have a $\rho df_\bot$ proof tree from $\{(p, \mathsf{sc}, b), (b, \mathsf{sc}, f), (p, \mathsf{sc}, e), (e, \bot_{\mathsf{c}}, f)\}$ to $(p, \bot_{\mathsf{c}}, p)\}$.

By Definitions 2.3 and 3.9, the following propositions is immediate to prove.

Proposition 3.4. *Let G be a $\rho df_\bot$ graph and (s, p, o) be a $\rho df_\bot$ triple. Then $G \vdash_{\rho df_\bot} (s, p, o)$ iff there is a $\rho df_\bot$ proof tree from H to (s, p, o) for some $H \subseteq G$.*

As next we define a depth function on the trees.

Definition 3.10 (Immediate subtree and depth function d)**.** *Let T be a $\rho df_\bot$ proof tree. The set of the* immediate subtrees *of T, $\mathfrak{T} = \{T_1, \ldots, T_n\}$, are the trees T_i obtained from T by eliminating the root node of T.*

The depth $d(T) \in \mathbb{N}$ *of T is defined inductively the following way:*

- *if T is a single node then $d(T) = 0$;*
- *else, $d(T) = 1 + \max\{d(T') \mid T' \in \mathfrak{T}\}$.*

Now, we prove the following lemmas.

Lemma 3.5. *Let T be a $\rho df_\bot$ proof tree from H to (p, sc, q). Then T contains only triples of the form (A, sc, B).*

Proof. The proof is on induction on the depth $d(T)$ of T, where by assumption, T has (p, sc, q) as root.

Case $d(T) = 0$**.** Hence, the tree's only node is (p, sc, q), which concludes.

Case $d(T) = 1$**.** In this case, there is only one possible tree, obtained by instantiating rule (3a):

$$\frac{(p, \mathsf{sc}, r), (r, \mathsf{sc}, q)}{(p, \mathsf{sc}, q)}$$

Remark 3.2. *Note that we cannot instantiate rule (2b) in the form*

$$\frac{(r, \mathsf{sp}, \mathsf{sc}), (p, r, q)}{(p, \mathsf{sc}, q)},$$

as $(r, \mathsf{sp}, \mathsf{sc})$ *is not allowed to occur in our language (see Section 2).*

Case $d(T) = n + 1$**.** Let us assume that the lemma holds for all proof trees of depth $m \leqslant n$, with $n \geqslant 1$. Let us show that it holds also for the case $d(T) = n + 1$ as well.

Note that the tree T of depth $n + 1$ with root (p, sc, q) can only be built by taking two trees T_1 and T_2 that have as roots triples of the form (A_i, sc, B_i) ($i = 1, 2$) with $\max(d(T_1), d(T_2)) = n$, and applying to their roots rule (3a). Therefore, by construction of T and by induction on T_i, also the tree T of depth $n + 1$ contains only triples of the form (A, sc, B), which concludes.

□

Lemma 3.6. *Let* T *be a* $\rho df_\perp$ *proof tree from* H *to* $(p, \perp_\mathsf{c}, q)$*. Then* T *contains only triples of the form* (A, sc, B) *or* $(A, \perp_\mathsf{c}, B)$*.*

Proof. The proof is on induction on the depth $d(T)$ of T, where by assumption, T has $(p, \perp_\mathsf{c}, q)$ as root.

Case $d(T) = 0$**.** Hence, the tree's only node is $(p, \perp_\mathsf{c}, q)$, which concludes.

Case $d(T) = 1$**.** In this case, there are only three possible trees, obtained by instantiating rules $(5a)$, $(5b)$ or $(5c)$: namely,

$$\frac{(q, \perp_\mathsf{c}, p)}{(p, \perp_\mathsf{c}, q)}, \frac{(s, \perp_\mathsf{c}, q), (p, \mathsf{sc}, s)}{(p, \perp_\mathsf{c}, q)} \text{ or } \frac{(p, \perp_\mathsf{c}, p)}{(p, \perp_\mathsf{c}, q)}.$$

In all three cases the lemma is satisfied, which concludes.

Case $d(T) = n + 1$**.** Let us assume that the lemma holds for all proof trees of depth $m \leqslant n$, with $n \geqslant 1$. Let us show that it holds also for the case $d(T) = n + 1$ as well.

Note that the tree T of depth $n + 1$ with root $(p, \perp_\mathsf{c}, q)$ can only be built in three ways:

- by applying rule (5a) to a tree T_1 of depth n having as root a triple of form $(A, \perp_\mathsf{c}, B)$;
- by applying the rule (5b) to two trees T_2 and T_3, with $\max(d(T_2), d(T_3)) = n$, having as root, respectively, a triple of form $(A, \perp_\mathsf{c}, B)$ and a triple of form (A, sc, B);
- by applying rule (5c) to a tree T_4 of depth n having as root a triple of form $(A, \perp_\mathsf{c}, A)$;

Now, by construction of T, by induction hypothesis on T_1, T_2, T_4 and by Lemma 3.5 applied to T_3, also the tree T of depth $n + 1$ contains only triples of the form (A, sc, B) and $(A, \perp_\mathsf{c}, B)$, which concludes.

□

Note that a tree proving $(t, \perp_\mathsf{c}, t)$ is just a particular case of Lemma 3.6.

Lemma 3.7. *Let G be a defeasible $\rho df_\perp$-graph, $\mathcal{R} = (\mathscr{M}, r)$ be a ranked model in $\mathfrak{R}_G$ and $\langle p, \mathsf{sc}, q\rangle \in G^{def}$. For every $\mathcal{I} \in \mathscr{M}$ s.t. $r(\mathcal{I}) = 0$, either $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{sc}, q)$ or $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_\mathsf{c}, p)$.*

Analogously, for every $\langle p, \mathsf{sp}, q\rangle \in G^{def}$ and every $\mathcal{I} \in \mathscr{M}$ s.t. $r(\mathcal{I}) = 0$, either $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{sp}, q)$ or $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_\mathsf{p}, p)$.

Proof. The proof is immediate from the fact that every $\mathcal{R}$ in $\mathfrak{R}_G$ is a model of G and Definition 3.4. In fact, if there is $\mathcal{I} \in \mathscr{M}$ s.t. $r(\mathcal{I}) = 0$ and $\mathcal{I}$ does satisfy neither (p, sc, q) nor $(p, \perp_\mathsf{c}, p)$, then $\mathcal{R} \nVdash_{\rho df_\perp} \langle p, \mathsf{sc}, q\rangle$ and, thus, $\mathcal{R}$ is not a model of G, against the hypothesis.

The proof for $\langle p, \mathsf{sp}, q\rangle \in G^{def}$ is similar. □

We can extend the above lemma to derived subclass and subproperty triples.

Lemma 3.8. *Let G be a defeasible $\rho df_\perp$-graph, $\mathcal{R} = (\mathscr{M}, r)$ be a ranked model in $\mathfrak{R}_G$ and let $G^s \vdash_{\rho df_\perp} (p, \mathsf{sc}, q)$ for some terms p, q. For every $\mathcal{I} \in \mathscr{M}$ s.t. $r(\mathcal{I}) = 0$, either $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{sc}, q)$ or $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_\mathsf{c}, p)$.*

Analogously, if $G^s \vdash_{\rho df_\perp} (p, \mathsf{sp}, q)$ for some terms p, q, then for every $\mathcal{I} \in \mathscr{M}$ s.t. $r(\mathcal{I}) = 0$, either $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{sp}, q)$ or $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_\mathsf{p}, p)$.

Proof. We prove the first half, involving the subclass predicate.

So, assume $G^s \vdash_{\rho df_\perp} (p, \mathsf{sc}, q)$ and let $\mathcal{I}$ be a $\rho df_\perp$ interpretation in $\mathscr{M}$ that has rank 0. The proof is on induction on the depth $d(T)$ of a tree T, where T has (p, sc, q) as root.

Case $d(T) = 0$**.** Hence, the tree's only node is (p, sc, q). Therefore, either $(p, \mathsf{sc}, q) \in G^{str}$ or $\langle p, \mathsf{sc}, q\rangle \in G^{def}$. In the former case $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{sc}, q)$, as, being $\mathcal{R}$ a model of G, every $\mathcal{I} \in \mathscr{M}$ must satisfy G^{str}. In the latter case, by Lemma 3.7, either $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{sc}, q)$ or $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_\mathsf{c}, p)$, which concludes.

Case $d(T) = 1$**.** In this case, there is only one possible tree, obtained by instantiating rule (3a):

$$\frac{(p, \mathsf{sc}, r), (r, \mathsf{sc}, q)}{(p, \mathsf{sc}, q)}$$

Assume $\mathcal{I} \nVdash_{\rho df_\perp} (p, \mathsf{sc}, q)$. As sc is a transitive relation, we have two possibilities only:

- **Case** $\mathcal{I} \nVdash_{\rho df_\perp} (p, \mathsf{sc}, r)$**.** Then, since $(p, \mathsf{sc}, r) \in G^s$, it must be the case that $\langle p, \mathsf{sc}, r\rangle \in G^{def}$ and, thus, by Lemma 3.7, $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_\mathsf{c}, p)$.
- **Case** $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{sc}, r)$ **but** $\mathcal{I} \nVdash_{\rho df_\perp} (r, \mathsf{sc}, q)$**.** Then, since $(r, \mathsf{sc}, q) \in G^s$, it must be the case that $\langle r, \mathsf{sc}, q\rangle \in G^{def}$. By Lemma 3.7, $\mathcal{I} \Vdash_{\rho df_\perp} (r, \perp_\mathsf{c}, r)$. So we have $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{sc}, r)$ and $\mathcal{I} \Vdash_{\rho df_\perp} (r, \perp_\mathsf{c}, r)$. Given the derivation rule (EmptySC') and the fact that $\vdash_{\rho df_\perp}$ is sound, we can conclude that $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_\mathsf{c}, p)$.

Therefore, either $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{sc}, q)$ or $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_\mathsf{c}, p)$, which concludes.

Case $d(T) = n + 1$**.** Let us assume that the lemma holds for all proof trees of depth $m \leqslant n$, with $n \geqslant 1$. Let us show that it holds also for the case $d(T) = n + 1$ as well, where T has (p, sc, q) as root. Now, since $d(T) > 1$, as for case $d(T) = 1$, the only possibility is that the tree terminates with an instantiation of rule (3a): that is,

$$\frac{(p, \mathsf{sc}, r), (r, \mathsf{sc}, q)}{(p, \mathsf{sc}, q)}$$

where (p, sc, r) and (r, sc, q) are, respectively, the roots of trees T_1 and T_2, the immediate subtrees of T, with $\max(d(T_1), d(T_2)) = n$. By inductive hypothesis on T_i, either $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{sc}, r)$ or $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_\mathsf{c}, p)$, and $\mathcal{I} \Vdash_{\rho df_\perp} (r, \mathsf{sc}, q)$ or $\mathcal{I} \Vdash_{\rho df_\perp} (r, \perp_\mathsf{c}, r)$.

As a consequence, we have three cases:

Case $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{c}}, p)$**.** The lemma is satisfied immediately.

Case $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{sc}, r)$ **and** $\mathcal{I} \Vdash_{\rho df_\perp} (r, \perp_{\mathsf{c}}, r)$**.** Then, given the derivation rule (EmptySC') and the fact that $\vdash_{\rho df_\perp}$ is sound, we have $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{c}}, p)$, which concludes.

Case $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{sc}, r)$ **and** $\mathcal{I} \Vdash_{\rho df_\perp} (r, \mathsf{sc}, q)$**.** Then, by the transitivity of the interpretation of sc, we have $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{sc}, q)$, which concludes.

For the second half of the lemma, involving the predicate sp, the proof has exactly the same structure, with the only difference that it refers to the rule (2a) instead of the rule (3a). □

Lemma 3.9. *Let G be a defeasible graph, p and q terms, and let $\mathcal{R}_{\min G} = (\mathscr{M}, r)$ be G's minimal model. If $G^s \vdash_{\rho df_\perp} (p, \perp_{\mathsf{c}}, q)$ then $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{c}}, q)$ for every $\mathcal{I} \in \mathscr{M}$ s.t. $r(\mathcal{I}) = 0$.*

Proof. Let G be a defeasible graph s.t. $G^s \vdash_{\rho df_\perp} (p, \perp_{\mathsf{c}}, q)$. Then by Proposition 3.4, there is a $\rho df_\perp$ proof tree T from H to $(p, \perp_{\mathsf{c}}, q)$ for some $H \subseteq G^s$. We prove now the lemma by induction on the depth $d(T)$ of T.

Given $\mathcal{R}_{\min G} = (\mathscr{M}, r)$, we need to prove that $(p, \perp_{\mathsf{c}}, q)$ is satisfied by every $\mathcal{I} \in \mathscr{M}$ s.t. $r(\mathcal{I}) = 0$. Note that, as $\mathcal{R}_{\min G}$ is a model of G, for every $\mathcal{I} \in \mathscr{M}$, $\mathcal{I} \Vdash_{\rho df_\perp} G^{str}$ holds (see Definition 3.4).

Case $d(T) = 0$**.** The tree's only node is $(p, \perp_{\mathsf{c}}, q)$, that is in G^s. Recall that G^s is the union of G^{str} and the strict translation of the defeasible triples in G^{def} (Equation 1). Note that $(p, \perp_{\mathsf{c}}, q)$ cannot be the strict form of a triple in G^{def}, as the defeasible triples must contain sc or sp as second element (Definition 3.1), hence $(p, \perp_{\mathsf{c}}, q) \in G^{str}$. Being $\mathcal{R}_{\min G}$ a model of G, its strict part G^{str} must be satisfied by every $\mathcal{I} \in \mathscr{M}$ and, as a consequence, the triple $(p, \perp_{\mathsf{c}}, q)$ must be satisfied by every $\mathcal{I} \in \mathscr{M}$ with rank 0, which concludes.

Case $d(T) = 1$**.** There are only three possible trees of depth 1 with a triple $(p, \perp_{\mathsf{c}}, q)$ obtained by instantiating rules $(5a)$, $(5b)$ or $(5c)$:

$$\frac{(q, \perp_{\mathsf{c}}, p)}{(p, \perp_{\mathsf{c}}, q)}, \frac{(s, \perp_{\mathsf{c}}, q), (p, \mathsf{sc}, s)}{(p, \perp_{\mathsf{c}}, q)} \text{ or } \frac{(p, \perp_{\mathsf{c}}, p)}{(p, \perp_{\mathsf{c}}, q)} .$$

In the first case, we can refer to the case $d(T) = 0$ and the fact that $\mathcal{I}$ must satisfy the symmetry of $\perp_{\mathsf{c}}$. The third case is similar to the first one taking into account that $\mathcal{I}$ must be c-exhaustive on $\perp_{\mathsf{c}}$. In the second case the tree consists of an instantiation of rule (5b), that has two premises, $(s, \perp_{\mathsf{c}}, q)$ and (p, sc, s), that must both be in G^s. $(s, \perp_{\mathsf{c}}, q)$ does not have a correspondent defeasible triple, so $(s, \perp_{\mathsf{c}}, q) \in G^{str}$. Since G^{str} must be satisfied by every ρdf-interpretation in a ranked model of G, we have $\mathcal{I} \Vdash_{\rho df_\perp} (s, \perp_{\mathsf{c}}, q)$.

Concerning the premise (p, sc, s), we have two possible cases:

Case $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{sc}, s)$**.** In this case, from $\mathcal{I} \Vdash_{\rho df_\perp} (s, \perp_{\mathsf{c}}, q)$ and the soundness of $\vdash_{\rho df_\perp}$, we can conclude $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{c}}, q)$.

Case $\mathcal{I} \not\Vdash_{\rho df_\perp} (p, \mathsf{sc}, s)$**.** In this case, $(p, \mathsf{sc}, s) \in G^s$ and $\mathcal{I} \not\Vdash_{\rho df_\perp} (p, \mathsf{sc}, s)$ implies that $(p, \mathsf{sc}, s) \notin G^{str}$, so $\langle p, \mathsf{sc}, s \rangle \in G^{def}$ must hold. By Lemma 3.7, $\mathcal{I} \not\Vdash_{\rho df_\perp} (p, \mathsf{sc}, s)$, implies that $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{c}}, p)$. Now, by rule $(5c)$ and the soundness of $\vdash_{\rho df_\perp}$, we can conclude $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{c}}, q)$.

Case $d(T) = n + 1$**.** Let us assume that the proposition holds for all the proof trees of depth $m \leqslant n$, with $n \geqslant 1$. Let us show that it holds also for the case $d(T) = n + 1$ as well, where T has $(p, \perp_{\mathsf{c}}, q)$ as root. Then, since $d(T) > 1$, as for the case $d(T) = 1$ the only three possibilities are that the tree terminates with an application rule $(5a)$, $(5b)$ or $(5c)$: that is,

$$\frac{(q, \perp_{\mathsf{c}}, p)}{(p, \perp_{\mathsf{c}}, q)}, \frac{(s, \perp_{\mathsf{c}}, q), (p, \mathsf{sc}, s)}{(p, \perp_{\mathsf{c}}, q)} \text{ or } \frac{(p, \perp_{\mathsf{c}}, p)}{(p, \perp_{\mathsf{c}}, q)} .$$

The first case is straightforward, since the immediate subtree of T would be a tree of depth n that has $(q, \bot_{\mathsf{c}}, p)$ as root: by inductive hypothesis $\mathcal{I} \Vdash_{\rho df_\bot} (q, \bot_{\mathsf{c}}, p)$ and, since $\mathcal{I}$ must satisfy the symmetry of $\bot_{\mathsf{c}}$, we have $\mathcal{I} \Vdash_{\rho df_\bot} (p, \bot_{\mathsf{c}}, q)$. The third case can be proven similarly to the first case by relying on fact that $\mathcal{I}$ is c-exhaustive over Δ_{C}. In the second case $(s, \bot_{\mathsf{c}}, q)$ and (p, sc, s) are, respectively, the roots of trees T_1 and T_2, the immediate subtrees of T, with $\max(d(T_1), d(T_2)) = n$. By inductive hypothesis on T_1, $\mathcal{I} \Vdash_{\rho df_\bot} (s, \bot_{\mathsf{c}}, q)$. Concerning T_2 and its root (p, sc, s) we have two possible cases:

Case $\mathcal{I} \Vdash_{\rho df_\bot} (p, \mathsf{sc}, s)$**.** In this case, by rule (5b) and the soundness of $\vdash_{\rho df_\bot}$ we conclude $\mathcal{I} \Vdash_{\rho df_\bot} (p, \bot_{\mathsf{c}}, q)$.

Case $\mathcal{I} \nVdash_{\rho df_\bot} (p, \mathsf{sc}, s)$**.** In this case, by the second part of Lemma 3.8 we have $\mathcal{I} \Vdash_{\rho df_\bot} (p, \bot_{\mathsf{c}}, p)$. Now, by rule $(5c)$ and the soundness of $\vdash_{\rho df_\bot}$, we can conclude $\mathcal{I} \Vdash_{\rho df_\bot} (p, \bot_{\mathsf{c}}, q)$.

This concludes the proof of then lemma. □

The following is an immediate corollary of Lemma 3.9.

Corollary 3.10. *For any a defeasible graph G and any term t, if $G^s \vdash_{\rho df_\bot} (t, \bot_{\mathsf{c}}, t)$ then t is* c*-exceptional w.r.t. G.*

Now we prove the analogous result for p-exceptionality.

Lemma 3.11. *Let G be a defeasible graph, p, q be any pair of terms, and let $\mathcal{R}_{\min G} = (\mathscr{M}, r)$ be G's minimal model. If $G^s \vdash_{\rho df_\bot} (p, \bot_{\mathsf{p}}, q)$ then $\mathcal{I} \Vdash_{\rho df_\bot} (p, \bot_{\mathsf{p}}, q)$ for every $\mathcal{I} \in \mathscr{M}$ s.t. $r(\mathcal{I}) = 0$.*

Proof. Let G be a defeasible graph s.t. $G^s \vdash_{\rho df_\bot} (p, \bot_{\mathsf{p}}, q)$. Then by Proposition 3.4 there must be a $\rho df_\bot$ proof tree T deriving $(p, \bot_{\mathsf{p}}, q)$ from some graph $H \subseteq G^s$. We prove now the lemma by induction on the depth $d(T)$ of T.

Given $\mathcal{R}_{\min G} = (\mathscr{M}, r)$, we need to prove that $(p, \bot_{\mathsf{p}}, q)$ is satisfied by every $\mathcal{I} \in \mathscr{M}$ s.t. $r(\mathcal{I}) = 0$. Note that, as $\mathcal{R}_{\min G}$ is a model of G, for every $\mathcal{I} \in \mathscr{M}$, $\mathcal{I} \Vdash_{\rho df_\bot} G^{str}$ holds (see Definition 3.4).

Case $d(T) = 0$**.** The tree's only node is $(p, \bot_{\mathsf{p}}, q)$, that is in G^s. Recall that G^s is the union of G^{str} and the strict translation of the defeasible triples in G^{def} (Equation 1). Note that $(p, \bot_{\mathsf{p}}, q)$ cannot be the strict form of a triple in G^{def}, since the defeasible triples must contain sc or sp as second element (Definition 3.1), hence $(p, \bot_{\mathsf{p}}, q) \in G^{str}$. Being $\mathcal{R}_{\min G}$ a model of G, its strict part G^{str} must be satisfied by every $\mathcal{I} \in \mathscr{M}$ (see Definition 3.4), and as a consequence the triple $(p, \bot_{\mathsf{p}}, q)$ must be satisfied by every $\mathcal{I} \in \mathscr{M}$ with rank 0, which concludes.

Case $d(T) = 1$**.** There are five possible trees of depth 1 with a triple $(p, \bot_{\mathsf{p}}, q)$ as root, obtained by instantiating rule $(6a)$, $(6b)$, $(6c)$, $(7a)$ or $(7b)$: *i.e.*,

(1) $\frac{(q, \bot_{\mathsf{p}}, p)}{(p, \bot_{\mathsf{p}}, q)}$

(2) $\frac{(s, \bot_{\mathsf{p}}, q), (p, \mathsf{sp}, s)}{(p, \bot_{\mathsf{p}}, q)}$

(3) $\frac{(p, \bot_{\mathsf{p}}, p)}{(p, \bot_{\mathsf{p}}, q)}$

(4) $\frac{(p, \mathsf{dom}, r), (q, \mathsf{dom}, s), (r, \bot_{\mathsf{c}}, s)}{(p, \bot_{\mathsf{p}}, q)}$

(5) $\frac{(p, \mathsf{range}, r), (q, \mathsf{range}, s), (r, \bot_{\mathsf{c}}, s)}{(p, \bot_{\mathsf{p}}, q)}$.

Case (1). We can refer to the case $d(T) = 0$ and the fact that $\mathcal{I}$ must satisfy the symmetry of $\perp_{\mathsf{p}}$.

Case (3). This case is similar to Case (1) by referring to the fact that $\mathcal{I}$ is c-exhaustive.

Case (2). The tree consists of an instantiation of rule (6b), that has two premises, $(s, \perp_{\mathsf{p}}, q)$ and (p, sp, s), that must both be in G^s. $(s, \perp_{\mathsf{p}}, q)$ does not have a correspondent defeasible triple, so $(s, \perp_{\mathsf{p}}, q) \in G^s$ implies $(s, \perp_{\mathsf{p}}, q) \in G^{str}$. Since G^{str} must be satisfied by every ρdf-interpretation in a ranked model of G, we have $\mathcal{I} \Vdash_{\rho df_\perp} (s, \perp_{\mathsf{p}}, q)$.

Concerning, the premise (p, sp, s), we have two possible cases:

Case $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{sp}, s)$**.** In this case, from $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{sp}, s)$, $\mathcal{I} \Vdash_{\rho df_\perp} (s, \perp_{\mathsf{p}}, q)$, and the soundness of $\vdash_{\rho df_\perp}$, we can conclude $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{p}}, q)$.

Case $\mathcal{I} \not\Vdash_{\rho df_\perp} (p, \mathsf{sp}, s)$**.** In this case, the situation in which $(p, \mathsf{sp}, s) \in G^s$ and $\mathcal{I} \not\Vdash_{\rho df_\perp} (p, \mathsf{sp}, s)$ is possible only if $(p, \mathsf{sp}, s) \in G^{def}$. By Lemma 3.7, we have $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{p}}, p)$. Now, by rule $(6c)$ and the soundness of $\vdash_{\rho df_\perp}$ we can conclude $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{p}}, q)$.

Case (4). $(p, \mathsf{dom}, r), (q, \mathsf{dom}, s), (r, \perp_{\mathsf{c}}, s)$ are in G^s, and since they cannot have a defeasible version, they must be in G^{str} too. All the triples in G^{str} must be satisfied by every $\mathcal{I} \in \mathscr{M}$. This, together with the soundness of $\vdash_{\rho df_\perp}$, guarantees that $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{p}}, q)$.

Case (5). Exactly as Case (4), just consider the triples of form (A, range, B) instead of the triples (A, dom, B).

Case $d(T) = n + 1$**.** Let us assume that the proposition holds for all the proof trees of depth $m \leqslant n$, with $n \geqslant 1$. Let us show that it holds also for the case $d(T) = n + 1$ as well, where T has $(p, \perp_{\mathsf{p}}, q)$ as root. Then, since $d(T) > 1$, as for the case with $d(T) = 1$ there are five possibilities, with the tree terminating with an application of the rule $(6a)$, $(6b)$, $(6c)$, $(7a)$ or $(7b)$: namely,

(1) $\frac{(q,\perp_{\mathsf{p}},p)}{(p,\perp_{\mathsf{p}},q)}$

(2) $\frac{(s,\perp_{\mathsf{p}},q),(p,\mathsf{sp},s)}{(p,\perp_{\mathsf{p}},q)}$

(3) $\frac{(p,\perp_{\mathsf{p}},p)}{(p,\perp_{\mathsf{p}},q)}$

(4) $\frac{(p,\mathsf{dom},r),(q,\mathsf{dom},s),(r,\perp_{\mathsf{c}},s)}{(p,\perp_{\mathsf{p}},q)}$

(5) $\frac{(p,\mathsf{range},r),(q,\mathsf{range},s),(r,\perp_{\mathsf{c}},s)}{(p,\perp_{\mathsf{p}},q)}$.

Case (1). Straightforward, as the immediate subtree of T is a tree of depth n that has $(q, \perp_{\mathsf{p}}, p)$ as root: by inductive hypothesis $\mathcal{I} \Vdash_{\rho df_\perp} (q, \perp_{\mathsf{p}}, p)$ and, since $\mathcal{I}$ must satisfy the symmetry of $\perp_{\mathsf{p}}$, we have $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{p}}, q)$.

Case (3). The proof is as for Case (1) by referring to the fact that $\mathcal{I}$ is c-exhaustive.

Case (2). $(s, \perp_{\mathsf{p}}, q)$ and (p, sp, s) are, respectively, the roots of T_1 and T_2, the immediate subtrees of T, with $\max(d(T_1), d(T_2)) = n$. By inductive hypothesis on T_1, $\mathcal{I} \Vdash_{\rho df_\perp} (s, \perp_{\mathsf{p}}, q)$. Concerning T_2 and its root (p, sp, s), we have two possible cases:

Case $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{sp}, s)$**.** In this case, from $\mathcal{I} \Vdash_{\rho df_\perp} (s, \perp_{\mathsf{p}}, q)$ and by the soundness of $\vdash_{\rho df_\perp}$ we can conclude $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{p}}, q)$.

Case $\mathcal{I} \not\Vdash_{\rho df_\perp} (p, \mathsf{sp}, s)$**.** In this case, by Lemma 3.8 we have $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{p}}, p)$. Now, by rule $(6c)$ and the soundness of $\vdash_{\rho df_\perp}$ we can conclude $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{p}}, q)$.

Case (4). Please note that for every pair of terms p, q, $G^s \vdash_{\rho df_\perp} (p, \mathsf{dom}, q)$ holds only if $(p, \mathsf{dom}, q) \in G^s$: no triple of form (A, dom, B) can be derived, since there are no rules in $\rho df_\perp$that have triples of form (A, dom, B) as conclusions. The only possibility could be the rule (2b), by substituting E with dom, but in that case in the premises we would have a triple with dom in the third position, that is not acceptable in our language (see Section 2.1).

Since the triples (A, dom, B) do not have a defeasible version, $(p, \mathsf{dom}, q) \in G^s$ implies that $(p, \mathsf{dom}, q) \in G^{str}$, and consequently $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{dom}, q)$. It follows that in case the tree T terminates with an application of rule (7a), it will have three immediate subtrees: two trees will both have depth 0 and will consist, respectively, only of the nodes (p, dom, r) and (q, dom, s); the third subtree, called T', will have have $(r, \perp_\mathsf{c}, s)$ as root.

We know from Lemma 3.9 applied to T' that $\mathcal{I} \Vdash_{\rho df_\perp} (r, \perp_\mathsf{c}, s)$. So we have that $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathsf{dom}, r)$, $\mathcal{I} \Vdash_{\rho df_\perp} (q, \mathsf{dom}, s)$, $\mathcal{I} \Vdash_{\rho df_\perp} (r, \perp_\mathsf{c}, s)$, and, by the soundness of $\vdash_{\rho df_\perp}$, we can conclude $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_\mathsf{p}, q)$.

Case (5). The proof for this case is analogous to Case (4): it is sufficient to substitute dom with range.

This concludes the proof of the lemma. □

An immediate corollary of Lemma 3.11 is the following.

Corollary 3.12. *For any a defeasible graph G and any term t, if $G^s \vdash_{\rho df_\perp} (t, \perp_\mathsf{p}, t)$ then t is* p-*exceptional w.r.t. G.*

Now we are ready to prove Proposition 3.3.

Proof of Proposition 3.3.

$\Rightarrow$.) Let us show that if a term t is c-exceptional (resp., p-exceptional) w.r.t. a defeasible graph G, then $G^s \vdash_{\rho df_\perp} (t, \perp_\mathsf{c}, t)$ (resp., $G^s \vdash_{\rho df_\perp} (t, \perp_\mathsf{p}, t)$).

Let t be a term in G, and let t be c-exceptional (resp., p-exceptional) w.r.t. G. Then t is c-exceptional (resp., p-exceptional) w.r.t. $\mathcal{R}_{\min G} = (\mathscr{M}, r)$, that is, for every $\mathcal{I} \in \mathscr{M}$ s.t. $r(\mathcal{I}) = 0$, we have that $\mathcal{I} \Vdash_{\rho df_\perp} (t, \perp_\mathsf{c}, t)$ (resp., $\mathcal{I} \Vdash_{\rho df_\perp} (t, \perp_\mathsf{p}, t)$). Now, we need to prove that $G^s \vdash_{\rho df_\perp} (t, \perp_\mathsf{c}, t)$ (resp., $G^s \vdash_{\rho df_\perp} (t, \perp_\mathsf{p}, t)$).

Let $\mathcal{I}_{G^s}$ be the canonical model of G^s. By construction and Lemma 2.3, $\mathcal{I}_{G^s} \in \mathscr{M}$. It is now sufficient to prove that $r(\mathcal{I}_{G^s}) = 0$. Let us proceed by contradiction by assuming that $r(\mathcal{I}_{G^s}) > 0$, and let $\mathcal{R}' = (\mathscr{M}, r')$ be the ranked interpretation obtained from $\mathcal{R}_{\min G}$, with for every $\mathcal{I} \in \mathscr{M}$,

$$r'(\mathcal{I}) = \begin{cases} 0 & \text{if } \mathcal{I} = \mathcal{I}_{G^s} \\ r(\mathcal{I}) & \text{otherwise .} \end{cases}$$

We can easily check that $\mathcal{R}'$ is a model of $G = G^{str} \cup G^{def}$:

- $\mathcal{R}'$ is still a model of G^{str}, since the satisfaction of $\rho df_\perp$-triples is not affected by the ranking function.
- for every $\langle p, \mathsf{sc}, q\rangle \in G^{def}$, $\mathcal{I}_{G^s} \Vdash_{\rho df_\perp} (p, \mathsf{sc}, q)$ since $\mathcal{I}_{G^s}$ is a model of G^s. Now we have two cases:

  $\mathcal{I}_{G^s} \Vdash_{\rho df_\perp} (p, \perp_\mathsf{c}, p)$. In such a case, by Definition 3.4, $\mathcal{I}_{G^s} \notin \mathsf{c\_}\min(p, \mathcal{R}')$ and $\mathcal{I}_{G^s}$ is irrelevant to decide the satisfaction of $\langle p, \mathsf{sc}, q\rangle$ in $\mathcal{R}'$. Consequently $\mathsf{c\_}\min(p, \mathcal{R}') = \mathsf{c\_}\min(p, \mathcal{R}_{\min G})$, and $\mathcal{R}_{\min G} \Vdash_{\rho df_\perp} \langle p, \mathsf{sc}, q\rangle$ implies $\mathcal{R}' \Vdash_{\rho df_\perp} \langle p, \mathsf{sc}, q\rangle$.

Procedure `ExceptionalC`(G)

Input: Defeasible graph $G = G^{str} \cup G^{def}$

Output: Set $\epsilon^{\mathsf{c}}(G)$ of c-exceptional triples w.r.t. G

$\epsilon^{\mathsf{c}}(G) := \emptyset$

for all $\langle p, \mathtt{sc}, q\rangle \in G^{def}$ **do**

  if $G^s \vdash_{\rho df_\perp} (p, \perp_{\mathtt{c}}, p)$ **then**

    $\epsilon^{\mathsf{c}}(G) := \epsilon^{\mathsf{c}}(G) \cup \{\langle p, \mathtt{sc}, q\rangle\}$

return $\epsilon^{\mathsf{c}}(G)$

$\mathcal{I}_{G^s} \not\Vdash_{\rho df_\perp} (p, \perp_{\mathtt{c}}, p)$. Then, since $r'(\mathcal{I}_{G^s}) = 0$ and $r'(\mathcal{I}) = r(\mathcal{I})$ for all the other elements $\mathcal{I}$ of $\mathscr{M}$, $\mathcal{R}' \Vdash_{\rho df_\perp} \langle p, \mathtt{sc}, q\rangle$. Otherwise in $\mathcal{R}'$ there should be an $\mathcal{I}'$ with rank 0 s.t. $\mathcal{I}' \not\Vdash_{\rho df_\perp} (p, \mathtt{sc}, q)$ and $\mathcal{I}' \not\Vdash_{\rho df_\perp} (p, \perp_{\mathtt{c}}, p)$. But then also and $\mathcal{R}_{\min G}$ should contain such $\mathcal{I}'$ at rank 0, but that cannot be the case, as otherwise $\mathcal{R}_{\min G}$ would not satisfy $\langle p, \mathtt{sc}, q\rangle$, against the fact that $\mathcal{R}_{\min G}$ is a model of G.

Hence $\mathcal{R}' \Vdash_{\rho df_\perp} \langle p, \mathtt{sc}, q\rangle$.

- We proceed analogously to prove that for every $\langle p, \mathtt{sp}, q\rangle \in G^{def}$, $\mathcal{R}' \Vdash_{\rho df_\perp} \langle p, \mathtt{sp}, q\rangle$.

Therefore, $\mathcal{R}'$ is a model of G, but by Definition 3.5 we also have $\mathcal{R}' \prec \mathcal{R}_{\min G}$, against the definition of $\mathcal{R}_{\min G}$. Consequently it must be the case that $r(\mathcal{I}_{G^s}) = 0$, that implies that, for every t, $\mathcal{I}_{G^s} \Vdash_{\rho df_\perp} (t, \perp_{\mathtt{c}}, t)$ (resp., $\mathcal{I}_{G^s} \Vdash_{\rho df_\perp} (t, \perp_{\mathtt{p}}, t)$).

Since $\mathcal{I}_{G^s}$ is the canonical model for G^s, $G^s \vdash_{\rho df_\perp} (t, \perp_{\mathtt{c}}, t)$ (resp., $G^s \vdash_{\rho df_\perp} (t, \perp_{\mathtt{p}}, t)$).

$\Leftarrow$.) Let us show that given a defeasible graph G, if $G^s \vdash_{\rho df_\perp} (t, \perp_{\mathtt{c}}, t)$ (resp., $G^s \vdash_{\rho df_\perp} (t, \perp_{\mathtt{p}}, t)$), then a term t is c-exceptional (resp., p-exceptional) w.r.t. G.

This is an immediate consequence of Corollaries 3.10 and 3.12.

This completes the proof of Proposition 3.3. □

Proposition 3.3 gives us a correct and complete correspondence between the semantic notions of c-exceptionality and p-exceptionality with the $\rho df_\perp$ decision procedure $\vdash_{\rho df_\perp}$. This correspondence allows us then to compute all the c-exceptional triples and all the p-exceptional ones, as illustrated by the procedures `ExceptionalC`(G) and `ExceptionalP`(G), respectively, where the notion of exceptional triple is defined in the obvious way:

Definition 3.11 (Exceptional triple)**.** *Let G be a defeasible $\rho df_\perp$-graph. We say that a defeasible triple $\langle p, \mathtt{sc}, q\rangle \in G^{def}$ (resp. $\langle p, \mathtt{sp}, q\rangle \in G^{def}$) is* c-exceptional *(resp.* p-exceptional*) w.r.t. G if p is* c-*exceptional (resp.* p-exceptional*) w.r.t. G.*

Procedure `ExceptionalP`(G)

Input: Defeasible graph $G = G^{str} \cup G^{def}$

Output: Set $\epsilon^{\mathsf{p}}(G)$ of p-exceptional triples w.r.t. G

$\epsilon^{\mathsf{p}}(G) := \emptyset$

for all $\langle p, \mathtt{sp}, q\rangle \in G^{def}$ **do**

  if $G^s \vdash_{\rho df_\perp} (p, \perp_{\mathtt{p}}, p)$ **then**

    $\epsilon^{\mathsf{p}}(G) := \epsilon^{\mathsf{c}}(G) \cup \{\langle p, \mathtt{sp}, q\rangle\}$

return $\epsilon^{\mathsf{p}}(G)$

Procedures $\texttt{ExceptionalC}(G)$ and $\texttt{ExceptionalP}(G)$ correctly model exceptionality, as proved by the following immediate corollary of Proposition 3.3.

Corollary 3.13. *Given a defeasible graph G and a defeasible triple $\langle p, \texttt{sc}, q\rangle \in G^{def}$ (resp., $\langle p, \texttt{sp}, q\rangle \in G^{def}$), $\langle p, \texttt{sc}, q\rangle \in \epsilon^{\mathsf{c}}(G)$ (resp., $\langle p, \texttt{sp}, q\rangle \in \epsilon^{\mathsf{p}}(G)$) iff it is* c*-exceptional (resp.* p*-exceptional) w.r.t. G.*

3.4 The ranking procedure

Iteratively applied, the notions of c-exceptionality and p-exceptionality allow us to associate to every term, *i.e.*, to every defeasible triple, a rank value w.r.t. a defeasible graph G. Specifically, we introduce a ranking procedure, called $\texttt{Ranking}(G)$, that orders the defeasible information in G^{def} into a sequence $\texttt{D}_0, \ldots, \texttt{D}_n, \texttt{D}_\infty$ of sets $\texttt{D}_i$ of defeasible triples, with $n \geqslant 0$ and $\texttt{D}_\infty$ possibly empty. The procedure is shown below.

Procedure $\texttt{Ranking}(G)$

Input: Defeasible graph $G = G^{str} \cup G^{def}$
Output: ranking $\texttt{r}(G) = \{\texttt{D}_0, \ldots, \texttt{D}_n, \texttt{D}_\infty\}$

$\texttt{D}_0 := G^{def}$
$i := 0$
repeat
  $\texttt{D}_{i+1} := \texttt{ExceptionalC}(G^{str} \cup \texttt{D}_i) \cup \texttt{ExceptionalP}(G^{str} \cup \texttt{D}_i)$
  $i := i + 1$
until $\texttt{D}_i = \texttt{D}_{i+1}$
$\texttt{D}_\infty := \texttt{D}_i$
$\texttt{r}(G) := \{\texttt{D}_0, \ldots, \texttt{D}_{i-1}, \texttt{D}_\infty\}$
return $\texttt{r}(G)$

The ranking procedure is built on top of the ExceptionalC and ExceptionalP procedures, using $\rho df_\perp$decision steps only.

We next prove that the ranking procedure $\texttt{Ranking}(G)$ correctly mirrors the ranking of the defeasible information w.r.t. the height functions h^{c}_G and h^{p}_G. Specifically, we want to show (see Proposition 3.19 later on) that

- for $i < n$, $\langle p, \texttt{sc}, q\rangle \in \texttt{D}_i \setminus \texttt{D}_{i+1}$ iff $h^{\mathsf{c}}_G(p) = i$;
- for $i = n$, $\langle p, \texttt{sc}, q\rangle \in \texttt{D}_n \setminus \texttt{D}_\infty$ iff $h^{\mathsf{c}}_G(p) = n$,

and an analogous result w.r.t. h^{p}_G.

To do so, we need to introduce some preliminary constructions and lemmas.

To start with, let us note that the information in a graph $G = G^{str} \cup G^{def}$ is ranked from the semantical point of view by the minimal model $\mathcal{R}_{\min G} = (\mathscr{M}, r)$ and the height functions h^{c}_G and h^{p}_G defined on it. Let $\mathcal{R}^i_{\min G} = (\mathscr{M}^i, r^i)$ be the submodel of $\mathcal{R}_{\min G}$ obtained by eliminating all the $\rho df_\perp$ interpretations in $\mathscr{M}$ whose height is strictly less than i ($i \geqslant 0$). Specifically, given a graph $G = G^{str} \cup G^{def}$ and its minimal model $\mathcal{R}_{\min G} = (\mathscr{M}, r)$, $\mathcal{R}^i_{\min G} = (\mathscr{M}^i, r^i)$ is defined as follows:

- $\mathscr{M}^i := \{\mathcal{I} \in \mathscr{M} \mid r(\mathcal{I}) \geqslant i\}$;
- $r^i(\mathcal{I}) := r(\mathcal{I}) - i$, for every $\mathcal{I} \in \mathscr{M}^i$.

That is, $\mathcal{R}^i_{\min G}$ is obtained from $\mathcal{R}_{\min G}$ by eliminating all the interpretations that have a rank of $i - 1$ or less. We next show that the interpretation $\mathcal{R}^i_{\min G}$ models the defeasible information in the graph that has a higher level of exceptionality: the higher the value of i, the higher the level of exceptionality. That is, given a defeasible graph $G = G^{str} \cup G^{def}$, let

$$G^i := G^{str} \cup G_i^{def} \ , \tag{4}$$

where

$$G_i^{def} := \{\langle p, \mathtt{sc}, q\rangle \in G^{def} \mid h_G^{\mathsf{c}}(p) \geqslant i\} \cup \{\langle p, \mathtt{sp}, q\rangle \in G^{def} \mid h_G^{\mathsf{p}}(p) \geqslant i\} \ . \tag{5}$$

Then the following proves that $\mathcal{R}^i_{\min G}$ is indeed a model of G^i.

Lemma 3.14. *Let $G = G^{str} \cup G^{def}$ be a defeasible graph and $\mathcal{R}_{\min G} = (\mathscr{M}, r)$ its minimal ranked model. Then $\mathcal{R}^i_{\min G} = (\mathscr{M}^i, r^i)$ is a model of the subgraph $G^i = G^{str} \cup G_i^{def}$.*

Proof. $\mathcal{R}_{\min G} = (\mathscr{M}, r)$ is a model of G. Hence all the interpretations in $\mathscr{M}$ are models of G^{str} and, since $\mathscr{M}^i \subseteq \mathscr{M}$, also $\mathcal{R}^i_{\min G}$ satisfies G^{str}. Concerning the defeasible triples in G_i^{def}, we proceed by contradiction, assuming $\mathcal{R}^i_{\min G}$ is not a model of G^i. So, let $\langle p, \mathtt{sc}, q\rangle \in G_i^{def}$ and $\mathcal{R}^i_{\min G} \not\Vdash_{\rho df_\perp} \langle p, \mathtt{sc}, q\rangle$, that is, there is a $\rho df_\perp$-interpretation $\mathcal{I}$ s.t. $\mathcal{I} \in \mathsf{c\_}\min(p, \mathcal{R}^i_{\min G})$ and $\mathcal{I} \not\Vdash_{\rho df_\perp} \langle p, \mathtt{sc}, q\rangle$. Since $h_G^{\mathsf{c}}(p) \geqslant i$, $\mathsf{c\_}\min(p, \mathcal{R}^i_{\min G}) = \mathsf{c\_}\min(p, \mathcal{R}_{\min G})$, and consequently $\mathcal{R}_{\min G} \not\Vdash_{\rho df_\perp} \langle p, \mathtt{sc}, q\rangle$, against the assumption that $\mathcal{R}_{\min G}$ is the minimal model of G. The case $\langle p, \mathtt{sp}, q\rangle \in G_i^{def}$ is proved similarly, which concludes. □

Now, it turns out that the minimal model for G^i, that is built using the set of $\rho df_\perp$-interpretations $\mathscr{M}$, can easily be defined by extending $\mathcal{R}^i_{\min G}$. That is, let $\mathcal{R}^*_i = (\mathscr{M}, r^*_i)$ be a ranked interpretation where r^*_i is defined as

$$r^*_i(\mathcal{I}) = \begin{cases} r^i(\mathcal{I}) & \text{if } \mathcal{I} \in \mathscr{M}^i \\ 0 & \text{otherwise .} \end{cases}$$

The following holds.

Lemma 3.15. *Given a defeasible graph G, $\mathcal{R}^*_i$ is the minimal model of the subgraph G^i.*

Proof. At first, we prove that $\mathcal{R}^*_i$ is a model of G^i. So, let $\mathcal{R}_{\min G}$ be the minimal model of the graph G. From the definitions of $\mathcal{R}_{\min G}$, $\mathcal{R}^i_{\min G}$ and $\mathcal{R}^*_i$ it is clear that for every $\mathcal{I} \in \mathscr{M}$, $r^*_i(\mathcal{I}) < \infty$ iff $r(\mathcal{I}) < \infty$. Hence $\mathcal{R}^*_i \Vdash_{\rho df_\perp} G^{str}$. Now, let $\langle p, \mathtt{sc}, q\rangle \in G_i^{def}$. From the construction of G_i^{def}, $\mathcal{R}^i_{\min G}$ and R^*_i, we have that for $\langle p, \mathtt{sc}, q\rangle \in G_i^{def}$ it holds that $h_G^{\mathsf{c}}(p) \geqslant i$. Therefore, since for every $\mathcal{I} \in \mathscr{M} \setminus \mathscr{M}^i$ $r(\mathcal{I}) < i$, it must be the case that $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{c}}, p)$ and, thus, for all the ρdf-interpretations $\mathcal{I} \in \mathscr{M} \setminus \mathscr{M}^i$, $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{c}}, p)$.

A consequence, for every $\langle p, \mathtt{sc}, q\rangle \in G_i^{def}$, $\mathsf{c\_}\min(p, \mathcal{R}_{\min G}) = \mathsf{c\_}\min(p, \mathcal{R}^i_{\min G})$, and since $\mathcal{R}_{\min G} \Vdash_{\rho df_\perp} \langle p, \mathtt{sc}, q\rangle$ for every $\langle p, \mathtt{sc}, q\rangle \in G_i^{def}$, $\mathcal{R}^*_i \Vdash_{\rho df_\perp} \langle p, \mathtt{sc}, q\rangle$ has to hold too.

Analogously, for every $\langle p, \mathtt{sp}, q\rangle \in G_i^{def}$, we have that

- for all the ρdf-interpretations $\mathcal{I} \in \mathscr{M} \setminus \mathscr{M}^i$, $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{p}}, p)$;
- $\mathsf{p\_}\min(p, \mathcal{R}_{\min G}) = \mathsf{p\_}\min(p, \mathcal{R}^i_{\min G})$; and
- for every $\langle p, \mathtt{sp}, q\rangle \in G_i^{def}$, $\mathcal{R}^*_i \Vdash_{\rho df_\perp} \langle p, \mathtt{sp}, q\rangle$.

Therefore, $\mathcal{R}^*_i$ is a model of G^i.

As next, we have to prove that $\mathcal{R}^*_i$ is in fact the *minimal* model of G^i. To do so, we proceed by contradiction, by assuming that this is not the case. Then there is a model $\mathcal{R}' = (\mathscr{M}, r')$ of G^i s.t. for every $\mathcal{I} \in \mathscr{M}$, $r'(\mathcal{I}) \leqslant r^*(\mathcal{I})$, and there is an $\mathcal{I}' \in \mathscr{M}$ s.t. $r'(\mathcal{I}') < r^*(\mathcal{I}')$. Note that, since $r^*(\mathcal{I}) = 0$ for every $\mathcal{I} \in \mathscr{M} \setminus \mathscr{M}^i$, $\mathcal{I}' \in \mathscr{M}^i$ necessarily. We have to prove that such an $\mathcal{R}'$ cannot exist.

Given $\mathcal{R}_{\min G} = (\mathscr{M}, r)$ and $\mathcal{R}' = (\mathscr{M}, r')$, we build a ranked interpretation $\mathcal{R}^+ = (\mathscr{M}, r^+)$ defining r^+ in the following way:

$$r^+(\mathcal{I}) = \begin{cases} r'(\mathcal{I}) + i & \text{if } \mathcal{I} \in \mathscr{M}^i \\ r(\mathcal{I}) & \text{otherwise.} \end{cases}$$

That is,

- $r^+(\mathcal{I}) = r(\mathcal{I})$ for every $\mathcal{I} \in \mathscr{M} \setminus \mathscr{M}^i$;
- from the definitions of r^i, r' and r^+ we can conclude that $r^+(\mathcal{I}) \leqslant r(\mathcal{I})$ for every $\mathcal{I} \in \mathscr{M}^i$, and there is an $\mathcal{I}' \in \mathscr{M}^i$ s.t. $r^+(\mathcal{I}') < r(\mathcal{I}')$.

As a consequence, $\mathcal{R}^+ \prec \mathcal{R}_{\min G}$. Also, $\mathcal{R}^+$ is a model of G, because

- all the $\rho df_\perp$-interpretations $\mathcal{I} \in \mathscr{M}$ are models of G^{str}, hence $\mathcal{R}^+ \Vdash_{\rho df_\perp} G^{str}$;
- for all the defeasible triples $\langle p, \mathsf{sc}, q\rangle \in G^{def} \setminus G_i^{def}$, $\mathsf{c\_}\min(p, \mathcal{R}^+) = \mathsf{c\_}\min(p, \mathcal{R}_{\min G})$, and, since $\mathcal{R}_{\min G} \Vdash_{\rho df_\perp} \langle p, \mathsf{sc}, q\rangle$, we have $\mathcal{R}^+ \Vdash_{\rho df_\perp} \langle p, \mathsf{sc}, q\rangle$;
- analogously, for all the defeasible triples $\langle p, \mathsf{sp}, q\rangle \in G^{def} \setminus G_i^{def}$, $\mathsf{p\_}\min(p, \mathcal{R}^+) = \mathsf{p\_}\min(p, \mathcal{R}_{\min G})$, and, since $\mathcal{R}_{\min G} \Vdash_{\rho df_\perp} \langle p, \mathsf{sp}, q\rangle$, we have $\mathcal{R}^+ \Vdash_{\rho df_\perp} \langle p, \mathsf{sp}, q\rangle$;
- for all the defeasible triples $\langle p, \mathsf{sc}, q\rangle \in G_i^{def}$, $\mathsf{c\_}\min(p, \mathcal{R}^+) = \mathsf{c\_}\min(p, \mathcal{R}')$, and, since $\mathcal{R}' \Vdash_{\rho df_\perp} \langle p, \mathsf{sc}, q\rangle$, we have $\mathcal{R}^+ \Vdash_{\rho df_\perp} \langle p, \mathsf{sc}, q\rangle$;
- analogously, for all the defeasible triples $\langle p, \mathsf{sp}, q\rangle \in G_i^{def}$, $\mathsf{p\_}\min(p, \mathcal{R}^+) = \mathsf{p\_}\min(p, \mathcal{R}')$, and, since $\mathcal{R}' \Vdash_{\rho df_\perp} \langle p, \mathsf{sp}, q\rangle$, we have $\mathcal{R}^+ \Vdash_{\rho df_\perp} \langle p, \mathsf{sp}, q\rangle$.

Therefore, $\mathcal{R}^+$ is a model of G, which is impossible, as $\mathcal{R}_{\min G}$ is the minimal model of G and, thus, $\mathcal{R}_i^*$ is the minimal model of the subgraph G^i. □

The following lemma connects the height of a term with the notion of exceptionality in the models $\mathcal{R}_i^*$.

Lemma 3.16. *Let $G = G^{str} \cup G^{def}$ be a defeasible graph and let $\mathcal{R}_{\min G}$ be its minimal model, with $h(\mathcal{R}_{\min G}) = n$. For every $i \leqslant n$ and term p s.t. $h_G^{\mathsf{c}}(p) \geqslant i$ (resp., $h_G^{\mathsf{p}}(p) \geqslant i$) p is c-exceptional (resp., p-exceptional) w.r.t. $\mathcal{R}_{\min G}^i = (\mathscr{M}^i, r^i)$ iff it is c-exceptional (resp., p-exceptional) w.r.t. $\mathcal{R}_i^* = (\mathscr{M}, r_i^*)$.*

Proof. Let $i \leqslant n$, with $R_i^* = (\mathscr{M}, r^*)$ and $\mathcal{R}_{\min G}^i = (\mathscr{M}^i, r^i)$ be the models of G^i built as described above.

For every $\mathcal{I} \in \mathscr{M}_i$, $r^i(\mathcal{I}) = r_i^*(\mathcal{I})$. If $\mathcal{I} \in \mathscr{M} \setminus \mathscr{M}^i$ we have seen above that $r^*(\mathcal{I}) = 0$ and $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{c}}, p)$ for every term p s.t. $h_G^{\mathsf{c}}(p) \geqslant i$.

Given these facts, the following statements are equivalent:

- p is not c-exceptional w.r.t. $\mathcal{R}_{\min G}^i$;
- there is an $\mathcal{I} \in \mathscr{M}^i$ s.t. $r^i(\mathcal{I}) = 0$ and $\mathcal{I} \not\Vdash_{\rho df_\perp} (p, \perp_{\mathsf{c}}, p)$;
- there is an $\mathcal{I} \in \mathscr{M}^i$ s.t. $r_i^*(\mathcal{I}) = 0$ and $\mathcal{I} \not\Vdash_{\rho df_\perp} (p, \perp_{\mathsf{c}}, p)$;
- p is not c-exceptional w.r.t. $\mathcal{R}_i^*$.

Analogously, we can prove that if we consider a term p s.t. $h_G^{\mathsf{p}}(p) \geqslant i$, the following statements are equivalent:

- p is not p-exceptional w.r.t. $\mathcal{R}^i_{\min G}$;
- there is an $\mathcal{I} \in \mathscr{M}^i$ s.t. $r^i(\mathcal{I}) = 0$ and $\mathcal{I} \not\Vdash_{\rho df_\bot} (p, \bot_{\mathsf{p}}, p)$;
- there is an $\mathcal{I} \in \mathscr{M}^i$ s.t. $r^*_i(\mathcal{I}) = 0$ and $\mathcal{I} \not\Vdash_{\rho df_\bot} (p, \bot_{\mathsf{p}}, p)$;
- p is not p-exceptional w.r.t. $\mathcal{R}^*_i$,

which concludes the proof. □

The following lemma connects the height to the computation of exceptionality.

Lemma 3.17. *Let $G = G^{str} \cup G^{def}$ be a defeasible graph, and $\mathcal{R}_{\min G} = (\mathscr{M}, r)$ its minimal model, with $h(\mathcal{R}_{\min G}) = n$, and $\langle p, \mathtt{sc}, q\rangle \in G^{def}$. Then,*

- *for every $i < n$, $h^{\mathsf{c}}_G(p) \geqslant i + 1$ iff $\langle p, \mathtt{sc}, q\rangle \in \mathtt{ExceptionalC}(G^i)$;*
- *for $i = n$, $h^{\mathsf{c}}_G(p) = \infty$ iff $\langle p, \mathtt{sc}, q\rangle \in \mathtt{ExceptionalC}(G^i)$.*

Analogously, let $\langle p, \mathtt{sp}, q\rangle \in G^{def}$. Then,

- *for every $i < n$, $h^{\mathsf{p}}_G(p) \geqslant i + 1$ iff $\langle p, \mathtt{sp}, q\rangle \in \mathtt{ExceptionalP}(G^i)$.*
- *for $i = n$, $h^{\mathsf{p}}_G(p) = \infty$ iff $\langle p, \mathtt{sp}, q\rangle \in \mathtt{ExceptionalP}(G^i)$.*

Proof. Let $\langle p, \mathtt{sc}, q\rangle \in G^{def}$.

For $i < n$, the following statements are equivalent:

- $h^{\mathsf{c}}_G(p) \geqslant i + 1$;
- for all $\mathcal{I} \in \mathscr{M} \setminus \mathscr{M}^{i+1}$, $\mathcal{I} \Vdash_{\rho df_\bot} (p, \bot_{\mathsf{c}}, p)$;
- for all $\mathcal{I} \in \mathscr{M} \setminus \mathscr{M}^{i}$, $\mathcal{I} \Vdash_{\rho df_\bot} (p, \bot_{\mathsf{c}}, p)$ and p is c-exceptional w.r.t. $\mathcal{R}^i_{\min G}$;
- for all $\mathcal{I} \in \mathscr{M} \setminus \mathscr{M}^{i}$, $\mathcal{I} \Vdash_{\rho df_\bot} (p, \bot_{\mathsf{c}}, p)$ and p is c-exceptional w.r.t. R^*_i (by Lemma 3.16);
- p is c-exceptional w.r.t. G^i (by Lemma 3.15);
- $\langle p, \mathtt{sc}, q\rangle \in \mathtt{ExceptionalC}(G^i)$ (by Corollary 3.13).

For $i = n$, the following statements are equivalent:

- $h^{\mathsf{c}}_G(p) = \infty$;
- for all the $\mathcal{I} \in \mathscr{M} \setminus \mathscr{M}^{\infty}$, $\mathcal{I} \Vdash_{\rho df_\bot} (p, \bot_{\mathsf{c}}, p)$;
- for all the $\mathcal{I} \in \mathscr{M} \setminus \mathscr{M}^{n}$, $\mathcal{I} \Vdash_{\rho df_\bot} (p, \bot_{\mathsf{c}}, p)$ and p is c-exceptional w.r.t. $\mathcal{R}^n_{\min G}$;
- for all the $\mathcal{I} \in \mathscr{M} \setminus \mathscr{M}^{n}$, $\mathcal{I} \Vdash_{\rho df_\bot} (p, \bot_{\mathsf{c}}, p)$ and p is c-exceptional w.r.t. R^*_i (by Lemma 3.16);
- p is c-exceptional w.r.t. G^n (by Lemma 3.15);
- $\langle p, \mathtt{sc}, q\rangle \in \mathtt{ExceptionalC}(G^n)$ (by Corollary 3.13).

This proves the first half of the proposition.

For triples $\langle p, \mathtt{sp}, q\rangle \in G^{def}$ the proof is analogous, which concludes. □

We now move on to prove the correspondence between the ranking procedure and the height (the semantic ranking of the defeasible information).

Given a defeasible graph $G = G^{str} \cup G^{def}$, $G_i^{\mathtt{D}}$ ($i \geqslant 0$) is the subgraph of G defined as follows:

$$G_i^{\mathtt{D}} := G^{str} \cup \mathtt{D}_i \ , \tag{6}$$

where $\mathtt{D}_i$ is in the output of $\mathtt{Ranking}(G)$, *i.e.*, $\mathtt{D}_i \in \mathtt{r}(G)$. Our objective now is to prove that $G^i = G_i^{\mathtt{D}}$. To prove that for every i, $G^i = G_i^{\mathtt{D}}$, it is sufficient to prove that for every rank value $i \neq \infty$ a term t is exceptional w.r.t. $G_i^{\mathtt{D}}$ iff it is exceptional w.r.t. $\mathcal{R}^i_{\min G}$.

The following can be shown.

Lemma 3.18. *Let G be a defeasible graph, with n being the height of its minimal model. Then, for every $i \leqslant n$, $G^i = G_i^{\mathtt{D}}$.*

Proof. We prove it by induction on the value of i. If $i = 0$, then $G_0^{def} = G^{def} = \mathtt{D}_0$, that is, $G^0 = G_0^{\mathtt{D}}$. Now, assume that $G^i = G_i^{\mathtt{D}}$ holds for all $i < n$, which implies also $G_i^{def} = \mathtt{D}_i$. By Lemma 3.17, for every $\langle p, \mathtt{sc}, q\rangle \in G_i^{def}$, $\langle p, \mathtt{sc}, q\rangle \in G_{i+1}^{def}$ iff $\langle p, \mathtt{sc}, q\rangle \in \mathtt{ExceptionalC}(G^i)$ iff $\langle p, \mathtt{sc}, q\rangle \in \mathtt{ExceptionalC}(G_i^{\mathtt{D}})$, that in turn is equivalent to $\langle p, \mathtt{sc}, q\rangle \in \mathtt{D}_{i+1}$.

Analogously, by Lemma 3.17, for every $\langle p, \mathtt{sp}, q\rangle \in G_i^{def}$, $\langle p, \mathtt{sp}, q\rangle \in G_{i+1}^{def}$ iff $\langle p, \mathtt{sp}, q\rangle \in \mathtt{ExceptionalP}(G^i)$ iff $\langle p, \mathtt{sp}, q\rangle \in \mathtt{ExceptionalP}(G_i^{\mathtt{D}})$, that in turn is equivalent to $\langle p, \mathtt{sp}, q\rangle \in \mathtt{D}_{i+1}$, which concludes. □

Now we can state the main proposition for our ranking procedure.

Proposition 3.19. *Let $G = G^{str} \cup G^{def}$ be a defeasible graph, $\langle p, \mathtt{sc}, q\rangle$ (resp., $\langle p, \mathtt{sp}, q\rangle$) be in G^{def}, and $\mathtt{r}(G) = \{\mathtt{D}_0, \dots, \mathtt{D}_n, \mathtt{D}_\infty\}$ be the ranking obtained by $\mathtt{Ranking}(G)$. The following statements hold:*

- *for $i < n$, $\langle p, \mathtt{sc}, q\rangle \in \mathtt{D}_i \setminus \mathtt{D}_{i+1}$ iff $h_G^{\mathtt{c}}(p) = i$;*
- *for $i = n$, $\langle p, \mathtt{sc}, q\rangle \in \mathtt{D}_n \setminus \mathtt{D}_\infty$ iff $h_G^{\mathtt{c}}(p) = n$.*

Proof. Immediate consequence of Lemma 3.18. □

Another consequence of Lemma 3.18 is the following corollary, that will be useful later on.

Corollary 3.20. *Let $G = G^{str} \cup G^{def}$ be a defeasible graph, with $\mathtt{r}(G) = \{\mathtt{D}_0, \dots, \mathtt{D}_n, \mathtt{D}_\infty\}$ being the ranking obtained by $\mathtt{Ranking}(G)$, and let p be a term. The following statements hold:*

- *$h_G^{\mathtt{c}}(p) = 0$ if and only if $G_0^{\mathtt{D}} \nvdash_{\rho df_\perp} (p, \perp_{\mathtt{c}}, p)$;*
- *for every i, $0 < i \leqslant n$, $h_G^{\mathtt{c}}(p) = i$ if and only if $G_{i-1}^{\mathtt{D}} \vdash_{\rho df_\perp} (p, \perp_{\mathtt{c}}, p)$ and $G_i^{\mathtt{D}} \nvdash_{\rho df_\perp} (p, \perp_{\mathtt{c}}, p)$;*
- *$h_G^{\mathtt{c}}(p) = \infty$ if and only if $G_n^{\mathtt{D}} \vdash_{\rho df_\perp} (p, \perp_{\mathtt{c}}, p)$.*

Analogously,

- *$h_G^{\mathtt{p}}(p) = 0$ if and only if $G_0^{\mathtt{D}} \nvdash_{\rho df_\perp} (p, \perp_{\mathtt{p}}, p)$;*
- *for every i, $0 < i \leqslant n$, $h_G^{\mathtt{p}}(p) = i$ if and only if $G_{i-1}^{\mathtt{D}} \vdash_{\rho df_\perp} (p, \perp_{\mathtt{p}}, p)$ and $G_i^{\mathtt{D}} \nvdash_{\rho df_\perp} (p, \perp_{\mathtt{p}}, p)$;*
- *$h_G^{\mathtt{c}}(p) = \infty$ if and only if $G_n^{\mathtt{D}} \vdash_{\rho df_\perp} (p, \perp_{\mathtt{p}}, p)$.*

Proof. Immediate from Proposition 3.3, Lemma 3.17 and Lemma 3.18. □

3.5 Decision procedures for defeasible ρdf

In this section we present our main decision procedures. That is, given triples $\langle p, \mathtt{sc}, q\rangle$, $\langle p, \mathtt{sp}, q\rangle$ and (s, p, o) as queries, the procedures below decide whether they are or not minimally entailed by a defeasible graph G.

Remark 3.3. *Given a fixed defeasible graph G, we assume that its ranking $\mathtt{r}(G) = \{\mathtt{D}_0, \dots, \mathtt{D}_n, \mathtt{D}_\infty\} = \mathtt{Ranking}(G)$ has already been computed .*

We start with the case of $\rho df_\perp$-triples of the form (s, p, o). The procedure `StrictMinEntailment` below decides whether $G \models_{\min} (s, p, o)$ holds.

Procedure $\mathtt{StrictMinEntailment}(G, \mathtt{r}(G), (s, p, o))$

Input: Graph $G = G^{str} \cup G^{def}$, ranking $\mathtt{r}(G) = \{\mathtt{D}_0, \dots, \mathtt{D}_n, \mathtt{D}_\infty\}$, a $\rho df_\perp$-triple (s, p, o)
Output: `true` iff $G \models_{\min} (s, p, o)$

1: $G' := G^{str} \cup \{(p, \perp_{\mathtt{c}}, p) \mid \langle p, \mathtt{sc}, q\rangle \in \mathtt{D}_\infty\} \cup \{(p, \perp_{\mathtt{p}}, p) \mid \langle p, \mathtt{sp}, q\rangle \in \mathtt{D}_\infty\}$
2: **return** $G' \vdash_{\rho df_\perp} (s, p, o)$

The following lemma can be proved, which motivates the construction of graph G' in the procedure above.

Lemma 3.21. *Let $G = G^{str} \cup G^{def}$ be a defeasible graph, $\mathtt{r}(G) = \{\mathtt{D}_0, \dots, \mathtt{D}_n, \mathtt{D}_\infty\}$ its ranking, and $\mathcal{R}_{\min G} = (\mathscr{M}, r)$ its minimal model. Then $\mathcal{I} \in \mathscr{M}_{\mathbb{N}}$ iff $\mathcal{I} \Vdash_{\rho df_\perp} G^{str} \cup \{(p, \perp_{\mathtt{c}}, p) \mid \langle p, \mathtt{sc}, q\rangle \in \mathtt{D}_\infty\} \cup \{(p, \perp_{\mathtt{p}}, p) \mid \langle p, \mathtt{sp}, q\rangle \in \mathtt{D}_\infty\}$.*

Proof.

$\Rightarrow$) From the construction of $G = G^{str} \cup G^{def}$ and Proposition 3.19, it is obvious that if $\mathcal{I} \in \mathscr{M}_{\mathbb{N}}$ then $\mathcal{I} \Vdash_{\rho df_\perp} G^{str} \cup \{(p, \perp_{\mathtt{c}}, p) \mid \langle p, \mathtt{sc}, q\rangle \in \mathtt{D}_\infty\} \cup \{(p, \perp_{\mathtt{p}}, p) \mid \langle p, \mathtt{sp}, q\rangle \in \mathtt{D}_\infty\}$.

$\Leftarrow$) We proceed by contradiction. Assume there is a $\rho df_\perp$-interpretation $\mathcal{I}' \in \mathscr{M}$ s.t. $r(\mathcal{I}') = \infty$ and $\mathcal{I}' \Vdash_{\rho df_\perp} G^{str} \cup \{(p, \perp_{\mathtt{c}}, p) \mid \langle p, \mathtt{sc}, q\rangle \in \mathtt{D}_\infty\} \cup \{(p, \perp_{\mathtt{p}}, p) \mid \langle p, \mathtt{sp}, q\rangle \in \mathtt{D}_\infty\}$.

Consider $h(\mathcal{R}_{\min G})$ (see Definition 3.7), and let $\mathcal{R}' = (\mathscr{M}, r')$ be a ranked interpretation where r' is defined in the following way:

$$r'(\mathcal{I}) = \begin{cases} r(\mathcal{I}) & \text{if } \mathcal{I} \in \mathscr{M}_{\mathbb{N}} \\ n+1 & \text{if } \mathcal{I} = \mathcal{I}' \\ \infty & \text{otherwise.} \end{cases}$$

Informally, $\mathcal{R}'$ has been obtained from $\mathcal{R}_{\min G}$ simply by moving $\mathcal{I}'$ from rank ∞ to the top of $\mathscr{M}_{\mathbb{N}}$. Clearly $\mathcal{R}' \prec \mathcal{R}_{\min G}$, since for every $\mathcal{I} \in \mathscr{M}$, $r'(\mathcal{I}) \leqslant r(\mathcal{I})$, and $r'(\mathcal{I}') < r(\mathcal{I}')$.

It is easy to check that $\mathcal{R}'$ is a model of G: every model with a finite rank satisfies G^{str}; for every $\langle p, \mathtt{sc}, q\rangle \in \mathtt{D}_i$, for $i < \infty$, $\mathtt{c\_}\min(p, \mathcal{R}') = \mathtt{c\_}\min(p, \mathcal{R}_{\min G})$, and consequently $\mathcal{R}' \Vdash_{\rho df_\perp} \langle p, \mathtt{sc}, q\rangle$. Analogously for every $\langle p, \mathtt{sp}, q\rangle \in \mathtt{D}_i$, for $i < \infty$. For every $\langle p, \mathtt{sc}, q\rangle \in \mathtt{D}_\infty$, every model with a finite rank satisfies $(p, \perp_{\mathtt{p}}, p)$; analogously for every $\langle p, \mathtt{sc}, q\rangle \in \mathtt{D}_\infty$. Hence $\mathcal{R}'$ is a model of G and $\mathcal{R}' \prec \mathcal{R}_{\min G}$, against the assumption that $\mathcal{R}_{\min G}$ is the minimal model of G.

As a consequently for every $\mathcal{I}$, $\mathcal{I} \Vdash_{\rho df_\perp} G^{str} \cup \{(p, \perp_{\mathtt{c}}, p) \mid \langle p, \mathtt{sc}, q\rangle \in \mathtt{D}_\infty\} \cup \{(p, \perp_{\mathtt{p}}, p) \mid \langle p, \mathtt{sp}, q\rangle \in \mathtt{D}_\infty\}$ implies $\mathcal{I} \in \mathscr{M}_{\mathbb{N}}$.

□

The following theorem establishes correctness and completeness of the `StrictMinEntailment` procedure.

Theorem 3.22. *Consider a defeasible graph G and a $\rho df_\perp$-triple (s,p,o). Then*

$$G \models_{\min} (s,p,o) \text{ iff } \texttt{StrictMinEntailment}(G, \texttt{r}(G), (s,p,o)) \; .$$

Proof. By Lemma 3.21 and Theorem 2.1 we have that $G \models_{\min} (s,p,o)$ if and only if $G^{str} \cup \{(p, \perp_{\texttt{c}}, p) \mid \langle p, \texttt{sc}, q\rangle \in \texttt{D}_\infty\} \cup \{(p, \perp_{\texttt{p}}, p) \mid \langle p, \texttt{sp}, q\rangle \in \texttt{D}_\infty\} \vdash_{\rho df_\perp} (s,p,o)$. Therefore, the procedure `StrictMinEntailment` is correct and complete. □

We next consider triples of the form $\langle p, \texttt{sc}, q\rangle$. The decision procedure `DefMinEntailmentC` below decides whether $G \models_{\min} \langle p, \texttt{sc}, q\rangle$ holds.

Procedure $\texttt{DefMinEntailmentC}(G, \texttt{r}(G), \langle p, \texttt{sc}, q\rangle)$

Input: Graph $G = G^{str} \cup G^{def}$, ranking $\texttt{r}(G) = \{\texttt{D}_0, \ldots, \texttt{D}_n, \texttt{D}_\infty\}$, defeasible triple $\langle p, \texttt{sc}, q\rangle$
Output: `true` iff $G \models_{\min} \langle p, \texttt{sc}, q\rangle$

$i := 0$
$\texttt{D}_{n+1} := \texttt{D}_\infty$
repeat
  if $i \leqslant n$ **then**
    $G' := G^{str} \cup (\texttt{D}_i)^s$
    $j := i$
    $i := i + 1$
  else
    return `true`
until $G' \nvdash_{\rho df_\perp} (p, \perp_{\texttt{c}}, p)$
$\texttt{D}^p := \{\langle r, \texttt{sc}, s\rangle \mid \langle r, \texttt{sc}, s\rangle \in \texttt{D}_j \setminus \texttt{D}_{j+1}\}$
return $G^{str} \cup (\texttt{D}^p)^s \vdash_{\rho df_\perp} (p, \texttt{sc}, q)$

To establish the correctness and completeness of procedure `DefMinEntailmentC` we prove some lemmas beforehand.

Let us recall that by Lemma 3.5, every proof tree with a triple $(p, \texttt{sc}, q)$ as root contains only triples of the form $(A, \texttt{sc}, B)$. From it, the following can be proven.

Lemma 3.23. *Let T be a proof tree from a graph H to a triple $(p, \texttt{sc}, q)$, and let $\mathcal{I}$ be a model of H. If $\mathcal{I} \Vdash_{\rho df_\perp} (s, \perp_{\texttt{c}}, s)$ for some triple $(s, \texttt{sc}, o) \in H$, then $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\texttt{c}}, p)$.*

Proof. The proof proceeds by induction on the depth of the proof tree T.

Case $d(T) = 0$. In this case, $H = \{(p, \texttt{sc}, q)\}$. Consequently $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\texttt{c}}, p)$ obviously implies $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\texttt{c}}, p)$.

Case $d(T) = 1$. The only way of deriving a triple of form $(p, \texttt{sc}, q)$ is using the rule (3a), as shown in the proof of Lemma 3.5. Hence the proof tree consist of an instantiation of rule (3a) with $(p, \texttt{sc}, q)$ as root.

$$(3a) \quad \frac{(p,\texttt{sc},t),(t,\texttt{sc},q)}{(p,\texttt{sc},q)}$$

Now, assume that $\mathcal{I} \Vdash_{\rho df_\perp} (A, \perp_{\texttt{c}}, A)$ holds for the antecedent of at least one of the two premises, *i.e.*, $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\texttt{c}}, p)$ or $\mathcal{I} \Vdash_{\rho df_\perp} (t, \perp_{\texttt{c}}, t)$ holds. We have to prove that $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\texttt{c}}, p)$.

Case (1). $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathtt{c}}, p)$. The result is immediate.

Case (2). $\mathcal{I} \Vdash_{\rho df_\perp} (t, \perp_{\mathtt{c}}, t)$. Then together with $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathtt{sc}, t)$ we derive $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathtt{c}}, p)$, by the soundness of rule (5b).

Case $d(T) = n + 1$**.** Assume that the proposition holds for all the proof trees with depth $m \leqslant n$, with $n > 1$. Let us show that it holds also for trees of depth $n + 1$. So, let T be a proof tree from H to (p, sc, q) with depth $n + 1$. The last step in T must correspond to an instantiation of the rule (3a) with (p, sc, q) as root:

$$(3a) \quad \frac{(p,\mathsf{sc},t),(t,\mathsf{sc},q)}{(p,\mathsf{sc},q)}$$

Hence T has two immediate subtrees: T', having (p, sc, t) as root, and T'', having (t, sc, q) as root; each of them has a depth of at most n. Since we assume that the proposition holds for trees of depth at most n, if $\mathcal{I} \Vdash_{\rho df_\perp} (s, \perp_{\mathtt{c}}, s)$ for some $(s, \mathtt{sc}, o) \in H$, then by induction hypothesis either $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathtt{c}}, p)$ (if $(s, \mathtt{sc}, o)$ appears as a leaf in T') or $\mathcal{I} \Vdash_{\rho df_\perp} (t, \perp_{\mathtt{c}}, t)$ (if $(s, \mathtt{sc}, o)$ appears as a leaf in T''). Likewise case $d(T) = 1$, in both the cases we can conclude $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathtt{c}}, p)$, which concludes the proof.

□

Lemma 3.24. *Let $G = G^{str} \cup G^{def}$ be a defeasible graph, and let $\mathtt{r}(G) = \{\mathtt{D}_0, \ldots, \mathtt{D}_n, \mathtt{D}_\infty\}$ be its ranking. For any pair of terms p, q s.t. $h^{\mathtt{c}}_G(p) \leqslant n$,*

$$G \models_{\min} \langle p, \mathtt{sc}, q\rangle \textit{ iff } G^{str} \cup (\mathtt{D}^p)^s \vdash_{\rho df_\perp} (p, \mathtt{sc}, q)\ .$$

where $\mathtt{D}^p$ is defined as in the `DefMinEntailmentC` *procedure.*

Proof. Consider $G = G^{str} \cup G^{def}$ and let $\mathcal{R}_{\min G} = (\mathscr{M}, r)$ be its minimal model. Let $h^{\mathtt{c}}_G(p) = k$, with $k \leqslant n$. From Corollary 3.20 we can conclude that in the the `DefMinEntailmentC` procedure, $G' \nvdash_{\rho df_\perp} (p, \perp_{\mathtt{c}}, p)$ for $j = k$, and $\mathtt{D}^p = \{\langle r, \mathtt{sc}, s\rangle \mid \langle r, \mathtt{sc}, s\rangle \in \mathtt{D}_k \setminus \mathtt{D}_{k+1}\}$.

We have to prove that $G \models_{\min} \langle p, \mathtt{sc}, q\rangle$ iff $G^{str} \cup (\mathtt{D}^p)^s \vdash_{\rho df_\perp} (p, \mathtt{sc}, q)$.

$\Leftarrow$.) Let us show that $G^{str} \cup (\mathtt{D}^p)^s \vdash_{\rho df_\perp} (p, \mathtt{sc}, q)$ implies $G \models_{\min} \langle p, \mathtt{sc}, q\rangle$. So, assume $G^{str} \cup (\mathtt{D}^p)^s \vdash_{\rho df_\perp} (p, \mathtt{sc}, q)$. To prove $G \models_{\min} \langle p, \mathtt{sc}, q\rangle$ we need to show that for every $\mathcal{I} \in \mathtt{c\_}\min(p, \mathcal{R}_{\min G})$, $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathtt{sc}, q)$ holds.

We proceed by contradiction. So, assume that $\mathcal{I} \in \mathtt{c\_}\min(p, \mathcal{R}_{\min G})$ and $\mathcal{I} \nVdash_{\rho df_\perp} (p, \mathtt{sc}, q)$. $\mathcal{I} \in \mathtt{c\_}\min(p, \mathcal{R}_{\min G})$ implies $\mathcal{I} \nVdash_{\rho df_\perp} (p, \perp_{\mathtt{c}}, p)$. Also, since $h^{\mathtt{c}}_G(p) = k$, $r(\mathcal{I}) = k$ follows. Since $G^{str} \cup (\mathtt{D}^p)^s \vdash_{\rho df_\perp} (p, \mathtt{sc}, q)$, $\mathcal{I} \nVdash_{\rho df_\perp} (p, \perp_{\mathtt{c}}, p)$ and $\mathcal{I} \Vdash_{\rho df_\perp} G^{str}$ (Lemma 3.21), then there is at least a triple in $(\mathtt{D}^p)^s$ that is not satisfied by $\mathcal{I}$. By Lemma 3.5 such a triple must be of the kind $(s, \mathtt{sc}, t)$. In particular, $G^{str} \cup (\mathtt{D}^p)^s \vdash_{\rho df_\perp} (p, \mathtt{sc}, q)$ implies that there must be a graph $H \subseteq G^{str} \cup (\mathtt{D}^p)^s$ s.t. there is a proof tree T proving $(p, \mathtt{sc}, q)$ from H, and there is some triple $(s, \mathtt{sc}, t) \in H \cap (\mathtt{D}^p)^s$ s.t. $\mathcal{I} \nVdash_{\rho df_\perp} (s, \mathtt{sc}, t)$. As $\langle s, \mathtt{sc}, t\rangle \in \mathtt{D}_k \setminus \mathtt{D}_{k+1}$, by Proposition 3.19 we have $h^{\mathtt{c}}_G(s) = k$. $h^{\mathtt{c}}_G(s) = k$ and $r(\mathcal{I}) = k$ imply that if $\mathcal{I} \nVdash_{\rho df_\perp} (s, \mathtt{sc}, t)$, then $\mathcal{I} \Vdash_{\rho df_\perp} (s, \perp_{\mathtt{c}}, s)$ ($\mathcal{I} \notin \mathtt{c\_}\min(s, \mathcal{R}_{\min G})$), otherwise $\mathcal{R}_{\min G}$ would not be a model of G. However, by Lemma 3.23, $\mathcal{I} \Vdash_{\rho df_\perp} (s, \perp_{\mathtt{c}}, s)$ implies $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathtt{c}}, p)$, against the hypothesis that $\mathcal{I} \in \mathtt{c\_}\min(p, \mathcal{R}_{\min G})$. Therefore, we must conclude that $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathtt{sc}, q)$.

$\Rightarrow$.) Let us show that $G \models_{\min} \langle p, \mathtt{sc}, q\rangle$ implies $G^{str} \cup (\mathtt{D}^p)^s \vdash_{\rho df_\perp} (p, \mathtt{sc}, q)$. So, let $h^{\mathtt{c}}_G(p) = k$, with $k \leqslant n$. $G \models_{\min} \langle p, \mathtt{sc}, q\rangle$ means that, given the minimal model $\mathcal{R}_{\min G} = (\mathscr{M}, r)$, for $\mathcal{I} \in \mathscr{M}$ s.t. $r(\mathcal{I}) = k$ and $\mathcal{I} \nVdash_{\rho df_\perp} (p, \perp_{\mathtt{c}}, p)$, $\mathcal{I} \in \mathtt{c\_}\min(p, \mathcal{R}_{\min G})$ and $\mathcal{I} \Vdash_{\rho df_\perp} (p, \mathtt{sc}, q)$.

Now, consider the graph $G^* := G^{str} \cup (\mathtt{D}_k \setminus \mathtt{D}_{k+1})^s \cup \{(r \bot_{\mathsf{c}} r) \mid \langle r, \mathtt{sc}, t\rangle \in \mathtt{D}_{k+1}\} \cup \{(r \bot_{\mathsf{p}} r) \mid \langle r, \mathtt{sp}, t\rangle \in \mathtt{D}_{k+1}\}$. That is, G^* contains all the strict triples in G, G^{str}, all the defeasible triples that are satisfied at height k, that is, $\mathtt{D}_i \setminus \mathtt{D}_{i+1}$, and for all the triples that are exceptional in k, the set $\{(r \bot_{\mathsf{c}} r) \mid \langle r, \mathtt{sc}, t\rangle \in \mathtt{D}_{k+1}\} \cup \{(r \bot_{\mathsf{p}} r) \mid \langle r, \mathtt{sp}, t\rangle \in \mathtt{D}_{k+1}\}$.

Let $\mathcal{I}_{G^*}$ be the characteristic $\rho df_\bot$ model of G^*, built as defined in Lemma 2.3. We know that $\mathcal{I}_{G^*}$ satisfies exactly $\mathsf{Cl}(G^*)$. Also, $\mathcal{I}_{G^*} \in \mathscr{M}$, as it can be checked by the definition in Section 3.2 of the ranked interpretations in $\mathfrak{R}_G$.

Since $\mathcal{I}_{G^*}$ satisfies $\mathcal{I} \Vdash_{\rho df_\bot} G^{str} \cup \{(r, \bot_{\mathsf{c}}, r) \mid \langle r, \mathtt{sc}, t\rangle \in \mathtt{D}_\infty\} \cup \{(r, \bot_{\mathsf{p}}, r) \mid \langle r, \mathtt{sp}, t\rangle \in \mathtt{D}_\infty\}$, by Lemma 3.21 it must be in $\mathscr{M}_\mathbb{N}$. Moreover, it must hold that $r(\mathcal{I}_{G^*}) = k$, since:

- for every triple $\langle s, \mathtt{sc}, t\rangle \in \mathtt{D}_k \setminus \mathtt{D}_{k+1}$ (resp., $\langle s, \mathtt{sp}, t\rangle \in \mathtt{D}_k \setminus \mathtt{D}_{k+1}$) $\mathcal{I}_{G^*}$ does not satisfy $(s, \bot_{\mathsf{c}}, s)$ (resp., $(s, \bot_{\mathsf{p}}, s)$), while $h^{\mathsf{c}}_G(s) = k$ (resp., $h^{\mathsf{p}}_G(s) = k$). Hence it cannot be $r(\mathcal{I}_{G^*}) < k$.
- $\mathcal{I}_{G^*}$ is compatible with $r(\mathcal{I}_{G^*}) = k$, since it satisfies all the triples in $\mathtt{D}_k \setminus \mathtt{D}_{k+1}$ and for every $\langle s, \mathtt{sc}, t\rangle \in \mathtt{D}_{k+1}$, $\mathcal{I}_{G^*} \Vdash_{\rho df_\bot} (s, \bot_{\mathsf{c}}, s)$ (resp., for every $\langle s, \mathtt{sp}, t\rangle \in \mathtt{D}_{k+1}$, $\mathcal{I}_{G^*} \Vdash_{\rho df_\bot} (s, \bot_{\mathsf{p}}, s)$).
- $\mathcal{R}_{\min G}$ assigns the minimal rank to every $\rho df_\bot$interpretation, and since $\mathcal{I}_{G^*}$ is compatible with $r(\mathcal{I}_{G^*}) = k$ and not with $r(\mathcal{I}_{G^*}) < k$, we can conclude $r(\mathcal{I}_{G^*}) = k$.

$h^{\mathsf{c}}_G(p) = k$ implies that $G^{\mathtt{D}}_k \not\models_{\rho df_\bot} (p, \bot_{\mathsf{c}}, p)$ (Corollary 3.20). Since $G^{\mathtt{D}}_k \models_{\rho df_\bot} G^*$, $G^* \not\models_{\rho df_\bot} (p, \bot_{\mathsf{c}}, p)$.

In summary, we have the following situation:

- $r(\mathcal{I}_{G^*}) = k$;
- $h^{\mathsf{c}}_G(p) = k$;
- since $G^* \not\models_{\rho df_\bot} (p, \bot_{\mathsf{c}}, p)$ and $\mathcal{I}_{G^*}$ is the characteristic model of G^*, $\mathcal{I}_{G^*} \not\Vdash_{\rho df_\bot} (p, \bot_{\mathsf{c}}, p)$.

Hence $\mathcal{I}_{G^*} \in \mathtt{c\_min}(p)$. This, together with $G \models_{\min} \langle p, \mathtt{sc}, q\rangle$, implies $\mathcal{I}_{G^*} \Vdash_{\rho df_\bot} (p, \mathtt{sc}, q)$. Being $\mathcal{I}_{G^*}$ the characteristic model of G^*, $\mathcal{I}_{G^*}$ satisfies a triple $(s, \mathtt{sc}, t)$ iff $(s, \mathtt{sc}, t) \in \mathsf{Cl}(G^*)$. Hence $\mathcal{I}_{G^*} \Vdash_{\rho df_\bot} (p, \mathtt{sc}, q)$ implies $G^* \vdash_{\rho df_\bot} (p, \mathtt{sc}, q)$.

By Lemma 3.5, we know that to derive a triple of form $(A, \mathtt{sc}, B)$ from a graph it is sufficient to consider only the triples in the same graph with the same form, and in G^* all such triples are in $G^{str} \cup (\mathtt{D}_k \setminus \mathtt{D}_{k+1})^s$, that is, $G^{str} \cup (\mathtt{D}^p)^s$. Hence $G^* \vdash_{\rho df_\bot} (p, \mathtt{sc}, q)$ implies

$$G^{str} \cup (\mathtt{D}^p)^s \vdash_{\rho df_\bot} (p, \mathtt{sc}, q) \ ,$$

which concludes.

□

The following theorem establishes correctness and completeness of the `DefMinEntailmentC` procedure.

Theorem 3.25. *Let $G = G^{str} \cup G^{def}$ be a defeasible graph and let $\langle p, \mathtt{sc}, q\rangle$ be a defeasible triple. Then,*

$$G \models_{\min} \langle p, \mathtt{sc}, q\rangle \textit{ iff } \mathtt{DefMinEntailmentC}(G, \mathtt{r}(G), \langle p, \mathtt{sc}, q\rangle) \ .$$

Proof. Let $\mathcal{R}_{\min G}$ be the minimal model of G, with $h(\mathcal{R}_{\min G}) = n$. Given $\langle p, \mathtt{sc}, q\rangle$ we have two possible cases:

Case $h^{\mathsf{c}}_G(p) \leqslant n$. The result is guaranteed by Lemma 3.24.

Case $h^{\mathsf{c}}_G(p) = \infty$. By Definition 3.4, $\mathcal{R}_{\min G} \Vdash_{\rho df_\bot} \langle p, \mathtt{sc}, q\rangle$. At the same time, if $h^{\mathsf{c}}_G(p) = \infty$, then by definition of the procedure, $\mathtt{DefMinEntailmentC}(G, \mathtt{r}(G), \langle p, \mathtt{sc}, q\rangle)$ must be the case.

□

Eventually, an analogous procedure to `DefMinEntailmentC` can be defined for the case of defeasible triples of the form $\langle p, \mathtt{sp}, q\rangle$, as illustrated by the `DefMinEntailmentP` procedure.

Procedure $\mathtt{DefMinEntailmentP}(G, \mathtt{r}(G), \langle p, \mathtt{sp}, q\rangle)$

Input: Graph $G = G^{str} \cup G^{def}$, ranking $\mathtt{r}(G) = \{\mathtt{D}_0, \dots, \mathtt{D}_n, \mathtt{D}_\infty\}$, defeasible triple $\langle p, \mathtt{sp}, q\rangle$
Output: `true` iff $G \models_{\min} \langle p, \mathtt{sp}, q\rangle$

$i := 0$
$\mathtt{D}_{n+1} := \mathtt{D}_\infty$
repeat
  if $i \leqslant n$ **then**
    $G' := G^{str} \cup (\mathtt{D}_i)^s$
    $j := i$
    $i := i + 1$
  else
    return `true`
until $G' \nvdash_{\rho df_\bot} (p, \bot_{\mathtt{p}}, p)$
$\mathtt{D}^p := \{\langle r, \mathtt{sp}, s\rangle \mid \langle r, \mathtt{sp}, s\rangle \in \mathtt{D}_j \setminus \mathtt{D}_{j+1}\}$
return $G^{str} \cup (\mathtt{D}^p)^s \vdash_{\rho df_\bot} (p, \mathtt{sp}, q)$

The proof that procedure `DefMinEntailmentP` is correct and complete w.r.t. $\models_{\min}$ proceeds similarly to the one for `DefMinEntailmentC`.

Specifically, we first prove the analogous of Lemma 3.5.

Lemma 3.26. *Let T be a $\rho df_\bot$ proof tree from H to $(p, \mathtt{sp}, q)$. Then T contains only triples of the form $(A, \mathtt{sp}, B)$.*

Proof. The proof is similar to the proof of Lemma 3.5, we just need to refer to rule (2a) instead of (3a). □

Also for the other propositions the proof is analogous to the correspondent propositions for the one for procedure `DefMinEntailmentC`. It suffices to change every instance of `sc` with `sp`, $\bot_{\mathtt{c}}$ with $\bot_{\mathtt{p}}$, every reference to Lemma 3.5 with Lemma 3.26, and so on. Specifically, we have

Lemma 3.27. *Let T be a proof tree from a graph H to a triple $(p, \mathtt{sp}, q)$, and let $\mathcal{I}$ be a model of H. If $\mathcal{I} \Vdash_{\rho df_\bot} (s, \bot_{\mathtt{p}}, s)$ for some triple $(s, \mathtt{sp}, o) \in H$, then $\mathcal{I} \Vdash_{\rho df_\bot} (p, \bot_{\mathtt{p}}, p)$.*

Lemma 3.28. *Let $G = G^{str} \cup G^{def}$ be a defeasible graph, and let $\mathtt{r}(G) = \{\mathtt{D}_0, \dots, \mathtt{D}_n, \mathtt{D}_\infty\}$ be its ranking. For any pair of terms p, q s.t. $h^{\mathsf{p}}_G(p) \leqslant n$,*

$$G \models_{\min} \langle p, \mathtt{sp}, q\rangle \textit{ iff } G^{str} \cup (\mathtt{D}^p)^s \vdash_{\rho df_\bot} (p, \mathtt{sp}, q)\ ,$$

where $\mathtt{D}^p$ is defined as in the `DefMinEntailmentP` *procedure.*

Eventually, we conclude with the following theorem establishing correctness and completeness of the `DefMinEntailmentP` procedure.

Theorem 3.29. *Let $G = G^{str} \cup G^{def}$ be a defeasible graph and let $\langle p, \mathtt{sp}, q\rangle$ be a defeasible triple. Then,*

$$G \models_{\min} \langle p, \mathtt{sp}, q\rangle \textit{ iff } \mathtt{DefMinEntailmentP}(G, \mathtt{r}(G), \langle p, \mathtt{sp}, q\rangle) \; .$$

We conclude this section with some examples.

Example 3.3. *Consider the graph H, similar to the graph F from Example 3.2, but with the following changes:*

- *the triple $\langle b, \mathsf{sc}, hf\rangle$ has been added, where hf is read as 'having feathers';*
- *The information that penguins do not fly has been made defeasible, that is, the triple (p, sc, e) has been substituted by $\langle p, \mathsf{sc}, e\rangle$;*
- *the triple $\langle pj, \mathsf{sc}, f\rangle$ has been added, where pj is read as 'penguins with jet-packs'.*

That is, $H = H^{str} \cup H^{def}$, with

$$H^{str} = \{(p, \mathsf{sc}, b), (r, \mathsf{sc}, b), (e, \bot_{\mathsf{c}}, f)\} \; ,$$

$$H^{def} = \{\langle b, \mathsf{sc}, f\rangle, \langle p, \mathsf{sc}, e\rangle, \langle b, \mathsf{sc}, hf\rangle, \langle pj, \mathsf{sc}, f\rangle\} \; .$$

Given $H^s = \{(p, \mathsf{sc}, b), (r, \mathsf{sc}, b), (b, \mathsf{sc}, f), (p, \mathsf{sc}, e), (e, \bot_{\mathsf{c}}, f), (b, \mathsf{sc}, hf), (pj, \mathsf{sc}, f)\}$, it is easy to check that:

- $H^s \nvdash_{\rho df_\bot} (b, \bot_{\mathsf{c}}, b)$;
- $H^s \nvdash_{\rho df_\bot} (r, \bot_{\mathsf{c}}, r)$;
- $H^s \vdash_{\rho df_\bot} (p, \bot_{\mathsf{c}}, p)$;
- $H^s \vdash_{\rho df_\bot} (pj, \bot_{\mathsf{c}}, pj)$.

That is, by procedure $\mathtt{Ranking}(H)$*,*

$$\mathtt{D}_1 = \{\langle p, \mathsf{sc}, e\rangle, \langle pj, \mathsf{sc}, f\rangle\} \; .$$

Since

- $H^{str} \cup \mathtt{D}_1{}^s \nvdash_{\rho df_\bot} (p, \bot_{\mathsf{c}}, p)$*, and*
- $H^{str} \cup \mathtt{D}_1{}^s \vdash_{\rho df_\bot} (pj, \bot_{\mathsf{c}}, pj)$*,*

the procedure $\mathtt{Ranking}(H)$ *gives back*

$$\mathtt{D}_2 = \{\langle pj, \mathsf{sc}, f\rangle\} \; ,$$

and terminates with

$$\mathtt{D}_\infty = \emptyset \; ,$$

as $H^{str} \cup \mathtt{D}_2{}^s \nvdash_{\rho df_\bot} (pj, \bot_{\mathsf{c}}, pj)$.

Given $\mathtt{Ranking}(H)$*, we can start checking what is minimally entailed. First of all, we can check that penguins do not fly, that is, we can make the following queries:*

Triple $\langle p, \mathsf{sc}, e\rangle$**.** *We apply the procedure* `DefMinEntailmentC`$(H, r(H), \langle p, \mathsf{sc}, e\rangle)$ *and have:*

$G' \not\vdash_{\rho df_\bot} (p, \bot_\mathsf{c}, p)$ *for* $i = 1$*, that implies* $\mathtt{D}^p = \{\langle p, \mathsf{sc}, e\rangle\}$.

$G^{str} \cup (\mathtt{D}^p)^s \vdash_{\rho df_\bot} (p, \mathsf{sc}, e)$*, since* $(\mathtt{D}^p)^s = \{(p, \mathsf{sc}, e)\}$*, and* `DefMinEntailmentC`$(H, r(H), \langle p, \mathsf{sc}, e\rangle)$ *returns* `true`.

Triple $\langle p, \mathsf{sc}, f\rangle$**.** *We apply the procedure* `DefMinEntailmentC`$(H, r(H), \langle p, \mathsf{sc}, f\rangle)$ *and have:*

$G^{str} \cup (\mathtt{D}^p)^s \not\vdash_{\rho df_\bot} (p, \mathsf{sc}, f)$*, and* `DefMinEntailmentC`$(H, r(H), \langle p, \mathsf{sc}, f\rangle)$ *returns* `false`.

The procedure `DefMinEntailmentC` *allows us to correctly derive that penguins do not fly, while it avoids to derive that penguins fly. What about the other typical property of birds in our graph, that is, having feathers?*

Triple $\langle p, \mathsf{sc}, hf\rangle$**.** *We apply the procedure* `DefMinEntailmentC`$(H, r(H), \langle p, \mathsf{sc}, hf\rangle)$ *and have:*

$G^{str} \cup (\mathtt{D}^p)^s \not\vdash_{\rho df_\bot} (p, \mathsf{sc}, hf)$*, and* `DefMinEntailmentC`$(H, r(H), \langle p, \mathsf{sc}, hf\rangle)$ *returns* `false`.

This latter example shows a well-known behaviour of RC: in the case a class is exceptional w.r.t. a super-class, it does not inherit any of the typical properties of the super-class. In this specific case, since penguins are exceptional birds (they do not fly), in RC they do not inherit any of the typical properties of birds: we cannot conclude that penguins have feathers. This behaviour, called the drowning effect *[15], may not be desirable in some applications. Let us recall that various RC extensions have been developed to overcome the drowning effect [57, 28, 26, 23, 48].*

Now we move to check the behaviour of sub-classes that do not show any exceptional behaviour. From our graph we know that robins are birds, and we have no information about any unusual property associated to robins. As a consequence, reasoning on the base of the principle of 'presumption of typicality' (see Section 3.2), we would like robins to inherit all the typical properties of birds. In fact, we have:

Triple $\langle r, \mathsf{sc}, f\rangle$**.** *We apply the procedure* `DefMinEntailmentC`$(H, r(H), \langle b, \mathsf{sc}, f\rangle)$ *and get:*

$G' \not\vdash_{\rho df_\bot} (r, \bot_\mathsf{c}, r)$ *for* $i = 0$*, that implies* $\mathtt{D}^p = \{\langle b, \mathsf{sc}, f\rangle, \langle b, \mathsf{sc}, hf\rangle\}$.

$G^{str} \cup (\mathtt{D}^p)^s \vdash_{\rho df_\bot} (r, \mathsf{sc}, f)$*, since* $(b, \mathsf{sc}, f) \in (\mathtt{D}^p)^s$ *and* $(r, \mathsf{sc}, b) \in G^{str}$*, it follows that* `DefMinEntailmentC`$(H, r(H), \langle r, \mathsf{sc}, f\rangle)$ *returns* `true`.

Triple $\langle r, \mathsf{sc}, hf\rangle$**.** *We apply the procedure* `DefMinEntailmentC`$(H, r(H), \langle b, \mathsf{sc}, hf\rangle)$ *and get:*

$G' \not\vdash_{\rho df_\bot} (r, \bot_\mathsf{c}, r)$ *for* $i = 0$.

$G^{str} \cup (\mathtt{D}^p)^s \vdash_{\rho df_\bot} (r, \mathsf{sc}, hf)$*, since* $(b, \mathsf{sc}, hf) \in (\mathtt{D}^p)^s$ *and* $(r, \mathsf{sc}, b) \in G^{str}$*, it follows that* `DefMinEntailmentC`$(H, r(H), \langle r, \mathsf{sc}, hf\rangle)$ *returns* `true`.

Eventually, we check what happens with an extra level of exceptionaly. Do penguins with jet-packs fly or not?

Triple $\langle pj, \mathsf{sc}, f\rangle$**.** *We apply the procedure* `DefMinEntailmentC`$(H, r(H), \langle pj, \mathsf{sc}, f\rangle)$ *and have:*

$G' \not\vdash_{\rho df_\bot} (p, \bot_\mathsf{c}, p)$ *for* $i = 2$*, that implies* $\mathtt{D}^p = \{\langle pj, \mathsf{sc}, f\rangle\}$.

$G^{str} \cup (\mathtt{D}^p)^s \vdash_{\rho df_\bot} (pj, \mathsf{sc}, f)$*, since* $(\mathtt{D}^p)^s = \{(pj, \mathsf{sc}, f)\}$*, and* `DefMinEntailmentC`$(H, r(H), \langle pj, \mathsf{sc} f\rangle)$ *returns* `true`.

Triple $\langle p, \mathsf{sc}, e\rangle$**.** *We apply the procedure* `DefMinEntailmentC`$(H, r(H), \langle pj, \mathsf{sc}, e\rangle)$ *and have:*

$G^{str} \cup (\mathtt{D}^p)^s \not\vdash_{\rho df_\bot} (p, \mathsf{sc}, e)$*, and* `DefMinEntailmentC`$(H, r(H), \langle p, \mathsf{sc}, e\rangle)$ *returns* `false`.

Therefore, correctly RC does not allow penguins with jet-packs to inherit the property of not-flying from typical penguins.

□

The next example shows a case in which we have some information with infinite rank, and how that indicates the presence of some conflict.

Example 3.4. *Let the graph* L *contain the following information*

$$L = \{(b, \mathsf{sc}, ba), (mb, \mathsf{sc}, b), (ba, \bot_{\mathsf{c}}, bw)\langle mb, \mathsf{sc}, bw\rangle\} ,$$

where ba *is read 'breaths air',* bw *is read 'breaths underwater', and* mb *is the class 'marsh bird'.*

When we apply the procedure $\texttt{Ranking}(L)$*, as*

$$L^s \vdash_{\rho df_\bot} (mb, \bot_{\mathsf{c}}, mb)$$

we obtain $\mathtt{D}_0 = \mathtt{D}_1 = \mathtt{D}_\infty = \{\langle mb, \mathsf{sc}, bw\rangle\}$*. We can check, using the procedure* $\texttt{StrictMinEntailment}$*, that* L *minimally entails* $(mb, \bot_{\mathsf{c}}, mb)$*: since* $\langle mb, \mathsf{sc}, bw\rangle \in \mathtt{D}_\infty$*, we have* $(mb, \bot_{\mathsf{c}}, mb) \in L'$*, that obviously imples*

$$L' \vdash_{\rho df_\bot} (mb, \bot_{\mathsf{c}}, mb) .$$

That is, $\texttt{StrictMinEntailment}(L, r(L), (mb, \bot_{\mathsf{c}}, mb))$ *returns* $\texttt{true}$*.*

The outcomes in Example 3.4 are reasonable and desirable: despite we are dealing with defeasible information, we are facing an unsolvable conflict: we are informed that birds breath air, (b, sc, ba), without exceptions, since it is a strict $\rho df_\bot$-triple, while marsh birds usually breath underwater, $\langle mb, \mathsf{sc}, bw\rangle$. The triple (b, sc, ba), not being defeasible, does not allow the existence of birds breathing underwater, and from the information at our disposal it is reasonable to conclude that marsh birds cannot exist, that is, $(mb, \bot_{\mathsf{c}}, mb)$.

Triples with infinite rank appear when there is some unsolvable conflict in the graph. Another example comes when we have in the graph pieces of information that are in direct conflict with each other. For example, if we add to the graph H in Example 3.3 the two defeasible triples $\langle mb, \mathsf{sc}, f\rangle, \langle mb, \mathsf{sc}, e\rangle$ ('marsch birds typically fly' and 'marsh birds typically do not fly'), that are in direct conflict with each other, we end up again with $(mb, \bot_{\mathsf{c}}, mb)$.

3.6 Structural properties

RC, like many other non-monotonic approaches, has also bee analysed from a 'structural properties' point of view [60]. For example, in the propositional case, given a knowledge base $\mathcal{K}$ of defeasible conditionals $\alpha \rightsquigarrow \beta$ (read 'if α holds, typically β holds too'), where α and β are propositions, RC satisfies a particular form of constrained monotonicity, called *Rational Monotonicity* (RM) [58]:

$$(RM) \quad \frac{\mathcal{K}\models\alpha\rightsquigarrow\beta, \;\; \mathcal{K}\not\models\alpha\rightsquigarrow\neg\gamma}{\mathcal{K}\models\alpha\wedge\gamma\rightsquigarrow\beta} \; .$$

The intended meaning of an instance of the structural property above is the following: if I know that typical birds fly ($\mathcal{K} \models \alpha \rightsquigarrow \beta$), and I am not aware that typical birds are not black ($\mathcal{K} \not\models \alpha \rightsquigarrow \neg\gamma$), then I may conclude that, typically, black birds fly ($\mathcal{K} \models \alpha \wedge \gamma \rightsquigarrow \beta$).

In our framework, a propositional defeasible conditional $\alpha \rightsquigarrow \beta$ correspond to defeasible triples of the form $\langle p, \mathtt{sc}, q\rangle$ and $\langle p, \mathtt{sp}, q\rangle$. Therefore, given a graph G, the property (RM) may take a form like

$$(RM) \quad \frac{G\models_{\rho df_\bot}\langle p,\mathtt{sc},q\rangle, \;\; G\not\models_{\rho df_\bot}\langle p,\mathtt{sc},\neg r\rangle}{G\models_{\rho df_\bot}\langle p\wedge r,\mathtt{sc},q\rangle} \; .$$

Such a property is linked to the use of conjunction and negation of terms, which, however, are not supported (so far) in $\rho df_\perp$. It does not seem possible to express (RM) in defeasible $\rho df_\perp$.

However, there are a few basic structural properties that still can be expressed in our framework. One of the simplest one is the property of *Supraclassicality*, as it is called in the propositional setting, that simply indicates that a strict piece of information implies also its own defeasible, weaker formulation, that is,

$$(Supra_c) \quad \frac{G \models_{\rho df_\perp} (p, \texttt{sc}, q)}{G \models_{\rho df_\perp} \langle p, \texttt{sc}, q \rangle}, \qquad (Supra_p) \quad \frac{G \models_{\rho df_\perp} (p, \texttt{sp}, q)}{G \models_{\rho df_\perp} \langle p, \texttt{sp}, q \rangle}$$

Proposition 3.30. $\models_{\rho df_\perp}$ *satisfies* $(Supra_c)$ *and* $(Supra_p)$.

Proof. Consider $(Supra_c)$, and let $G \models_{\rho df_\perp} (p, \texttt{sc}, q)$. That implies that all $\rho df_\perp$-interpretations in the minimal model $\mathcal{R}_{\min G}$ satisfy $(p, \texttt{sc}, q)$. Consequently, for every $\rho df_\perp$-interpretation $\mathcal{I}$ in $\texttt{c\_}\min(p, \mathcal{R}_{\min G})$, $\mathcal{I} \Vdash_{\rho df_\perp} (p, \texttt{sc}, q)$. That is, according to Definition 3.4, $\mathcal{R}_{\min G} \Vdash_{\rho df_\perp} \langle p, \texttt{sc}, q \rangle$, that is to $G \models_{\rho df_\perp} \langle p, \texttt{sc}, q \rangle$. The proof $(Supra_p)$ is analogous. □

Reflexivity, Left Logical Equivalence, and *Right Weakening* are other essential properties. In the propositional case they are:

$$(Ref) \quad \mathcal{K} \models \alpha \rightsquigarrow \alpha,$$

$$(LLE) \quad \frac{\mathcal{K} \models \alpha \rightsquigarrow \beta, \ \models \alpha \equiv \gamma}{\mathcal{K} \models \gamma \rightsquigarrow \beta}, \qquad (RW) \quad \frac{\mathcal{K} \models \alpha \rightsquigarrow \beta, \ \beta \models \gamma}{\mathcal{K} \models \alpha \rightsquigarrow \gamma},$$

As for *Supraclassicality*, these properties can also be translated in our system in two versions: one for classes and in one for predicates. For (LLE), logical equivalence '$\equiv$' is translated using symmetric pairs of sc- and sp-triples. Specifically, the above axioms are encoded as

$$(Ref_c) \quad G \models_{\rho df_\perp} \langle p, \texttt{sc}, p \rangle,$$

$$(LLE_c) \quad \frac{G \models_{\rho df_\perp} \langle p, \texttt{sc}, r \rangle, \ G \models_{\rho df_\perp} (p, \texttt{sc}, q), \ G \models_{\rho df_\perp} (q, \texttt{sc}, p)}{G \models_{\rho df_\perp} \langle q, \texttt{sc}, r \rangle}, \qquad (RW_c) \quad \frac{G \models_{\rho df_\perp} \langle p, \texttt{sc}, q \rangle, \ G \models_{\rho df_\perp} (q, \texttt{sc}, r)}{G \models_{\rho df_\perp} \langle p, \texttt{sc}, r \rangle},$$

$$(Ref_p) \quad G \models_{\rho df_\perp} \langle p, \texttt{sp}, p \rangle,$$

$$(LLE_p) \quad \frac{G \models_{\rho df_\perp} \langle p, \texttt{sp}, r \rangle, \ G \models_{\rho df_\perp} (p, \texttt{sp}, q), \ G \models_{\rho df_\perp} (q, \texttt{sp}, p)}{G \models_{\rho df_\perp} \langle q, \texttt{sp}, r \rangle}, \qquad (RW_p) \quad \frac{G \models_{\rho df_\perp} \langle p, \texttt{sp}, q \rangle, \ G \models_{\rho df_\perp} (q, \texttt{sp}, r)}{G \models_{\rho df_\perp} \langle p, \texttt{sp}, r \rangle},$$

Note that (Ref_c) and (Ref_p) do not hold, as they do not hold in the classical form $G \models_{\rho df_\perp} (p, \texttt{sc}, p)$ and $G \models_{\rho df_\perp} (p, \texttt{sp}, p)$. This is the consequence of having considered minimal ρdf for which reflexivity for sc and sp triples does not hold. If we use classical ρdf, we will recover reflexivity for the classical triples, and, by Proposition 3.30, we would obtain immediately also (Ref_c) and (Ref_p).

Concerning (LLE) and (RW), they are satisfied.

Proposition 3.31. $\models_{\rho df_\perp}$ *satisfies* (LLE_c), (RW_c), (LLE_p), *and* (RW_p).

Proof. We prove (LLE_c) and (RW_c). The proofs for (LLE_p) and (RW_i) are analogous.

(LLE_c). Let $G \models_{\rho df_\perp} (p, \texttt{sc}, q)$, $G \models_{\rho df_\perp} (q, \texttt{sc}, p)$,and $G \models_{\rho df_\perp} \langle p, \texttt{sc}, r\rangle$. $G \models_{\rho df_\perp} (p, \texttt{sc}, q)$ and $G \models_{\rho df_\perp} (q, \texttt{sc}, p)$ imply that every $\rho df_\perp$-intepretation in $\mathcal{R}_{\min G}$ satisfies $(p, \texttt{sc}, q)$ and $(q, \texttt{sc}, p)$. Since the semantics is sound w.r.t. the derivation rules, by rule $(5b)$ we have that for every $\rho df_\perp$-intepretation $\mathcal{I}$ in $\mathcal{R}_{\min G}$, $\mathcal{I} \Vdash_{\rho df_\perp} (p, \perp_{\mathsf{c}}, p)$ iff $\mathcal{I} \Vdash_{\rho df_\perp} (q, \perp_{\mathsf{c}}, q)$. Consequently $\texttt{c\_}\min(p, \mathcal{R}_{\min G}) = \texttt{c\_}\min(q, \mathcal{R}_{\min G})$.

Let $\mathcal{I}$ be in $\texttt{c\_}\min(p, \mathcal{R}_{\min G})$. Since $G \models_{\rho df_\perp} \langle p, \texttt{sc}, r\rangle$, $\mathcal{I} \Vdash_{\rho df_\perp} (p, \texttt{sc}, r)$. We also have $\mathcal{I} \Vdash_{\rho df_\perp} (q, \texttt{sc}, p)$, and, being $\mathcal{I}$ sound w.r.t. rule $(3a)$, we conclude $\mathcal{I} \Vdash_{\rho df_\perp} (q, \texttt{sc}, r)$. $\texttt{c\_}\min(p, \mathcal{R}_{\min G}) = \texttt{c\_}\min(q, \mathcal{R}_{\min G})$ implies $\mathcal{R}_{\min G} \Vdash_{\rho df_\perp} \langle q, \texttt{sc}, r\rangle$, that is, $G \models_{\rho df_\perp} \langle q, \texttt{sc}, r\rangle$.

(RW_c). $G \models_{\rho df_\perp} (q, \texttt{sc}, r)$ implies that every $\rho df_\perp$-intepretation in $\mathcal{R}_{\min G}$ satisfies $(q, \texttt{sc}, r)$. Let $\mathcal{I}$ be in $\texttt{c\_}\min(p, \mathcal{R}_{\min G})$. Since $G \models_{\rho df_\perp} \langle p, \texttt{sc}, q\rangle$, $\mathcal{I} \Vdash_{\rho df_\perp} (p, \texttt{sc}, q)$. Hence $\mathcal{I} \Vdash_{\rho df_\perp} (p, \texttt{sc}, r)$, that implies $\mathcal{R}_{\min G} \Vdash_{\rho df_\perp} \langle p, \texttt{sc}, r\rangle$, that is, $G \models_{\rho df_\perp} \langle p, \texttt{sc}, r\rangle$.

□

Remark 3.4. *Let us note that, despite (RM) seems not to be syntactically expressible in our framework, we have proposed the same kind of construction that is behind propositional RC, that is, we model a kind of defeasible reasoning implementing the* Presumption of Typicality *[59, p.4]: 'if we are not informed of the contrary, we reason assuming that we are dealing with typical behaviours. From a semantics point of view, this "maximisation of typicality" has been modelled considering the* minimal models *of a KB: those models in which the entities are associated to the lowest (i.e., most typical) possible rank value, modulo the satisfaction of the knowledge base. This formal solution has been used also elsewhere in order to give a semantic characterisation of RC [21, 49, 42, 68]. On the other hand, from the decision procedure point of view, we have defined a procedure that follows similar methods developed for RC in other formal frameworks; in particular, our decision procedure is built on top of the decision procedure for the monotonic fragment, and decides RC through a calculation of exceptionalities and rank values, as it is done e.g., for RC in the propositional and description logic frameworks [25, 30, 37].*

Nevertheless, there are some more properties satisfied by our entailment relation $\models_{\min}$. In the following, let us consider the closure operation $\mathsf{Cl}_{\models_{\min}}$ defined as follows: given a graph G,

$$\mathsf{Cl}_{\models_{\min}}(G) := \{[\![s, p, o]\!] \mid G \models_{\min} [\![s, p, o]\!]\}.$$

As we may expect, $\mathsf{Cl}_{\models_{\min}}$ is not monotonic, that is, given any graph G and triple $[\![s, p, o]\!]$, the following does not necessarily hold:

$$\mathsf{Cl}_{\models_{\min}}(G) \subseteq \mathsf{Cl}_{\models_{\min}}(G \cup \{[\![s, p, o]\!]\})$$

Proposition 3.32. $\mathsf{Cl}_{\models_{\min}}$ *is not monotonic.*

Proof. Consider the graph G in Example 3.1, and its subgraph

$$G' = \{(p, \mathsf{sc}, b), \langle b, \mathsf{sc}, f\rangle, (e, \perp_{\mathsf{c}}, f)\} ,$$

obtained eliminating the triple (p, sc, e).

Then it can be shown that w.r.t. G' penguins are not exceptional, *i.e.*,

$$\texttt{DefMinEntailmentP}(G', r(G'), \langle p, \mathsf{sc}, f\rangle) = \texttt{true} ,$$

while w.r.t. G penguins are exceptional, *i.e.*,

$$\texttt{DefMinEntailmentP}(G, r(G), \langle p, \mathsf{sc}, f\rangle) = \texttt{false} .$$

□

Nonetheless, and not surprising, $\mathsf{Cl}_{\models_{\min}}(G)$ is monotonic w.r.t. the strict part of the information.

Proposition 3.33. *Let G be any graph, (s,p,o) any $\rho df_\bot$-triple, and $[\![s',p',o']\!]$ be any triple. If $G \models_{\min} (s,p,o)$, then $G \cup \{[\![s',p',o']\!]\} \models_{\min} (s,p,o)$.*

Proof. This is immediate from Theorem 3.22. Indeed, $G \models_{\min} (s,p,o)$ implies $G' \vdash_{\rho df_\bot} (s,p,o)$, with G' defined according to the procedure `StrictMinEntailment`. Let $F := G \cup \{[\![s',p',o']\!]\}$, and let F' be defined applying the procedure `StrictMinEntailment` to F. Clearly $G' \subseteq F'$, and, given that $\vdash_{\rho df_\bot}$ is monotonic, we can conclude $F' \vdash_{\rho df_\bot} (s,p,o)$, that is, $F \models_{\min} (s,p,o)$. □

This is clearly a desirable behaviour, since the strict part of our information should be treated as non-defeasible information, and consequently we should reason monotonically about it.

$\mathsf{Cl}_{\models_{\min}}$ satisfies some other desirable properties: namely, *Inclusion, Cumulativity*, and *Idempotence*.

Proposition 3.34. $\mathsf{Cl}_{\models_{\min}}$ *satisfies inclusion. That is, for any graph G*

$$G \subseteq \mathsf{Cl}_{\models_{\min}}(G).$$

Proof. $\mathsf{Cl}_{\models_{\min}}(G)$ is determined by the model $\mathcal{R}_{\min G}$, and, by definition, $\mathcal{R}_{\min G} \in \mathfrak{R}_G$, that is, $\mathcal{R}_{\min G}$ is a model of G, implying $G \subseteq \mathsf{Cl}_{\models_{\min}}(G)$. □

Proposition 3.35. $\mathsf{Cl}_{\models_{\min}}$ *satisfies* Cumulativity. *That is, for any pair of graphs G, G',*

$$\text{If } G \subseteq G' \subseteq \mathsf{Cl}_{\models_{\min}}(G)\text{, then } \mathsf{Cl}_{\models_{\min}}(G') = \mathsf{Cl}_{\models_{\min}}(G).$$

Proof. $\mathsf{Cl}_{\models_{\min}}(G)$ is determined by the model $\mathcal{R}_{\min G}$, that is, the minimal model of G. Since $G' \subseteq \mathsf{Cl}_{\models_{\min}}(G)$, $\mathcal{R}_{\min G}$ is also a model of G'. Therefore, $\mathcal{R}_{\min G}$ must be the minimal model also for G'. Indeed, assume that is not the case, that is, there is a model R of G' such that $R \preceq \mathcal{R}_{\min G}$. Since $G \subseteq G'$, R is also a model of G, and $\mathcal{R}_{\min G}$ would not be the minimal model of G, against the hypothesis. Hence both $\mathsf{Cl}_{\models_{\min}}(G)$ and $\mathsf{Cl}_{\models_{\min}}(G')$ are determined by the model $\mathcal{R}_{\min G}$. □

Cumulativity has two immediate consequence that are well-known, desirable properties: a constrained form of monotonicity, called *Cautious Monotonicity*, and the classical property of *Cut*.

Proposition 3.36. $\mathsf{Cl}_{\models_{\min}}$ *satisfies* Cautious Monotonicity. *That is, for any graph G and any triple $[\![s,p,o]\!]$,*

$$\text{If } G \models_{\min} [\![s,p,o]\!] \text{ and } G \models_{\min} [\![s',p',o']\!]\text{, then } G \cup [\![s',p',o']\!] \models_{\min} [\![s,p,o]\!].$$

Proposition 3.37. $\mathsf{Cl}_{\models_{\min}}$ *satisfies* Cut. *That is, for any graph G and any triple $[\![s,p,o]\!]$,*

$$\text{If } G \cup [\![s',p',o']\!] \models_{\min} [\![s,p,o]\!] \text{ and } G \models_{\min} [\![s',p',o']\!]\text{, then } G \models_{\min} [\![s,p,o]\!].$$

Please note that these two properties are here analysed at the level of entailment. As discussed above, in the case of the conditional reasoning [58] these structural properties can be expressed at two levels: at the level of conditionals (see the first formulation of (RM) above), and the meta-level of the entailment relation. Here we are looking at the properties (CM) and (Cut) at the meta-level of the entailment relation. Similarly to the (RM) case, such properties at the level of the language may be expressed as

$$(CM) \quad \frac{G \models_{\rho df_\bot} \langle p, \mathsf{sc}, q\rangle,\ \ G \models_{\rho df_\bot} \langle p, \mathsf{sc}, r\rangle}{G \models_{\rho df_\bot} \langle p \wedge r, \mathsf{sc}, q\rangle}, \qquad (Cut) \quad \frac{G \models_{\rho df_\bot} \langle p \wedge r, \mathsf{sc}, q\rangle,\ \ G \models_{\rho df_\bot} \langle p, \mathsf{sc}, r\rangle}{G \models_{\rho df_\bot} \langle p, \mathsf{sc}, q\rangle},$$

if conjunction among terms would have been allowed.

Eventually, $\mathsf{Cl}_{\models_{\min}}$ satisfies also *Idempotence*, that is, $\mathsf{Cl}_{\models_{\min}}$ is actually a closure operation.

Proposition 3.38. $\mathsf{Cl}_{\models_{\min}}$ *satisfies* Idempotence. *That is, for any graph G,*

$$\mathsf{Cl}_{\models_{\min}}(\mathsf{Cl}_{\models_{\min}}(G)) = \mathsf{Cl}_{\models_{\min}}(G)\ .$$

Proof. Idempotence is an immediate consequence of *Inclusion* and *Cumulativity*. It is sufficient to set $G' = \mathsf{Cl}_{\models_{\min}}(G)$ in the Cumulativity property, while Inclusion guarantees that $G \subseteq \mathsf{Cl}_{\models_{\min}}(G)$ holds. □

3.7 Computational complexity

We now shortly address the computational complexity of the previously defined procedures and show that our entailment decision procedures run in polynomial time.

To start with, let us consider $\vdash_{\rho df_\perp}$. Now, consider a graph G and a $\rho df_\perp$-triple (s,p,q). An easy way to decide whether $G \vdash_{\rho df_\perp} (s,p,q)$ holds is to compute the closure $\mathsf{Cl}(G)$ of G and then check whether $(s,p,q) \in \mathsf{Cl}(G)$. Now, as G is ground, like for [63, Theorem 19], it is easily verified that, the size of the closure of G is $O(|G|^2)$ and, thus,

Proposition 3.39. *For a graph G and a $\rho df_\perp$-triple (s,p,q), $G \vdash_{\rho df_\perp} (s,p,q)$ can be decided in time $O(|G|^2)$.*

Remark 3.5. *Let us note that [63, Theorem 21] provides also an $O(|G|\log|G|)$ time algorithm to decide the ground ρdf entailment problem.*[10] *Whether a similar algorithm can be extended also to $\rho df_\perp$ (so including also rules (5)-(7)) while maintaining the same computational complexity is still an open problem.*

We next consider the `ExceptionalC` and `ExceptionalP` procedures. It is immediately verified that, by Proposition 3.5,

Proposition 3.40. *For a defeasible graph $G = G^{str} \cup G^{def}$, both* `ExceptionalC`(G) *and* `ExceptionalP`(G) *require at most $|G^{def}|$ $\vdash_{\rho df_\perp}$ checks and, thus, both run in time $O(|G^{def}||G|^2)$.*

Consider now the case of the `Ranking` procedure. It is easily verified that steps 3.-6. may be repeated at most $|G^{def}|$ times and each of which calls once the `ExceptionalC` and `ExceptionalP` procedures. Therefore, by Proposition 3.40, we have easily that

Proposition 3.41. *For a defeasible graph $G = G^{str} \cup G^{def}$,* `Ranking`(G) *runs in time $O(|G^{def}|^2|G|^2)$. Moreover, the number of sets in $\mathtt{r}(G)$ is at most $O(|G^{def}|)$.*

Eventually, let us consider the entailment check procedures in Section 3.5. Let us recall Remark 3.3 and, thus, we assume that the ranking has been computed once for all. The time required to compute $\mathtt{r}(G)$ is, by Proposition 3.41, $O(|G^{def}|^2|G|^2)$.

To what concerns `StrictMinEntailment`, the following result is an immediate from Proposition 3.39.

Proposition 3.42. *Consider a defeasible graph $G = G^{str} \cup G^{def}$ and a $\rho df_\perp$-triple (s,p,o). Then the procedure* `StrictMinEntailment`$(G, \mathtt{r}(G), (s,p,o))$ *runs in time $O(|G|^2)$.*

Now, consider `DefMinEntailmentC`. By Proposition 3.41, it is easily verified that the steps 3.-10. may be repeated at most $O(|G^{def}|)$ times and each time we make one $\vdash_{\rho df_\perp}$ check. By considering also the additional $\vdash_{\rho df_\perp}$ check in step 12, by Proposition 3.39 we have

Proposition 3.43. *Consider a defeasible graph $G = G^{str} \cup G^{def}$ and a defeasible triple $\langle p, \mathtt{sc}, q\rangle$. Then the procedure* `DefMinEntailmentC`$(G, \mathtt{r}(G), \langle p, \mathtt{sc}, q\rangle)$ *runs in time $O(|G^{def}||G|^2)$.*

The computational complexity of `DefMinEntailmentP` is the same as for `DefMinEntailmentC` and, thus, we conclude with

Proposition 3.44. *Consider a defeasible graph $G = G^{str} \cup G^{def}$ and a defeasible triple $\langle p, \mathtt{sp}, q\rangle$. Then the procedure* `DefMinEntailmentP`$(G, \mathtt{r}(G), \langle p, \mathtt{sc}, q\rangle)$ *runs in time $O(|G^{def}||G|^2)$.*

[10] It corresponds here to consider rules (1)-(4).

4 Brief Conclusions & Related Work

Contribution. We have shown how one may integrate RC within RDFS and, thus, obtain a non-monotone variant of the latter. To to do so, we started from ρdf, which is the logic behind RDFS, and then extended it to $\rho df_\bot$, allowing to state that two entities are incompatible. Eventually, we have worked out defeasible $\rho df_\bot$ via a typical RC construction.

The main features of our approach are summarised as follows:

- unlike other approaches that typically add an extra non-monotone rule layer on top of monotone RDFS (see below), defeasible $\rho df_\bot$ remains syntactically a triple language and is a simple vocabulary extension of ρdf by introducing some new predicate symbols, namely $\bot_\mathsf{c}$ and $\bot_\mathsf{p}$, with specific semantics allowing to state that two terms are incompatible. In particular, any RDFS reasoner/store may handle them as ordinary terms if it does not want to take account for the extra semantics of the new predicate symbols;
- the defeasible $\rho df_\bot$ entailment decision procedure is build on top of the $\rho df_\bot$ entailment decision procedure, which in turn is an extension of the one for ρdf via some additional inference rules favouring an potential implementation;
- we have shown that defeasible $\rho df_\bot$ entailment can be decided in polynomial time.

We have also analysed our proposal from a 'structural properties' point of view, describing which properties can be expressed in our proposal (not all can be expressed as Boolean connectives are missing in RDFS).

Related Work. There have been various works in the past about extending RDF/S with non-monotone capabilities, which we briefly summarise below.

The series of works [1, 2, 3, 4, 5, 6, 7, 31] essentially deal with *Extended* RDFS (ERDF), which consists in extending RDFS with two negations of Partial Logic [51], namely weak negation $\sim$ (expressing negation-as-failure or non-truth) and strong negation $\neg$ (expressing explicit negative information or falsity), as well as derivation rules. The semantics is based on some form of stable model semantics. So for instance, one may express the weak negation of a triple (s, p, o) as $\sim p(s, o)$, while its strong negation as $\neg p(s, o)$ and use them in rules, such as $\neg\mathsf{type}(x, EUMember) \leftarrow \mathsf{type}(x, AmericanCountry)$. Clearly, none of them can be represented in RDFS and there is also a computational complexity price to pay for, as illustrated in [7] (*e.g.*,. deciding model existence goes from NP to PSPACE, depending on the setting).

On a similar line, *i.e.*, using rule languages on top of RDFS, and borrowing non-monotone semantics developed within rules languages, are also all other approaches we are aware of, such as [8, 11, 12, 14, 16, 17, 18, 54, 55, 65] and practical large scale solutions such as [9, 10, 53, 73, 72, 77, 78]. Let us also note that there are more general solutions in providing a rule layer on top of ontology languages such as RDFS and OWL, which can be found in *e.g.*, [32, 33, 34, 35, 36].

Eventually, we refer to the related work section in [30] for a recent recap about works, such as *e.g.*, [22, 23, 27, 24, 25, 26, 29, 38, 40, 41, 42, 46, 45, 43, 44, 47, 20, 74, 70, 69, 71], about RC extension of DLs, the logic behind OWL 2 [66].

Future Work. Concerning future work, there are still some extension that are worth to be addressed within our framework. Two directions appear particularly promising: on one hand, as shown in Example 3.3, RC suffers of the *drowing effect*, and we would like to reformulate in the present framework some constructions defined for DL, such as [29, 26], that extend RC and overcome the drowning effect; on the other hand, we would like to extend defeasible triples also to other predicates of the ρdf vocabulary beyond `sc` and `sp`.

Another point deals with the computational complexity of our framework. In particular, we would like to see whether an approach similar as described in [63] can be applied to our context as well (see Remark 3.5).

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