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In this paper we show how to extend RDFS to express and reason with various forms of negative statements under the Open World Assumption (OWA). To do so, we start from rdf, a minimal, but significant RDFS fragment that covers all essential features of RDFS, and then extend it to ?rdfbotneg, allowing express also statements such as "radio therapies are non drug treatments", "Ebola has no treatment", or "opioids and antipyretics are disjoint classes". The main and, to the best of our knowledge, unique features of our proposal are: (i) rdfbotneg remains syntactically a triple language by extending rdf with new symbols with specific semantics and there is no need to revert to the reification method to represent negative triples; (ii) the logic is defined in such a way that any RDFS reasoner/store may handle the new predicates as ordinary terms if it does not want to take account of the extra capabilities; (iii) despite negated statements, every rdfbotneg knowledge base is satisfiable; (iv) the rdfbotneg entailment decision procedure is obtained from rdf via additional inference rules favouring a potential implementation; and (v) deciding entailment in rdfbotneg ranges from P to NP.

Keywords: Artificial Intelligence, Knowledge Representation, RDFS Graphs, Negative statements.

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A Minimal Deductive System for RDFS with Negative Statements

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Abstract

The triple language RDFS is designed to represent and reason with *positive* statements only (*e.g.*, “antipyretics are drugs”).

In this paper we show how to extend RDFS to express and reason with various forms of negative statements under the Open World Assumption (OWA). To do so, we start from ρdf, a minimal, but significant RDFS fragment that covers all essential features of RDFS, and then extend it to $\rho df_\bot^\neg$, allowing express also statements such as “radio therapies are *non* drug treatments”, “Ebola *has no* treatment”, or ”opioids and antipyretics are *disjoint* classes”. The main and, to the best of our knowledge, unique features of our proposal are: (i) $\rho df_\bot^\neg$ remains syntactically a triple language by extending ρdf with new symbols with specific semantics and there is no need to revert to the reification method to represent negative triples; (ii) the logic is defined in such a way that any RDFS reasoner/store may handle the new predicates as ordinary terms if it does not want to take account of the extra capabilities; (iii) despite negated statements, every $\rho df_\bot^\neg$ knowledge base is satisfiable; (iv) the $\rho df_\bot^\neg$ entailment decision procedure is obtained from ρdf via additional inference rules favouring a potential implementation; and (v) deciding entailment in $\rho df_\bot^\neg$ ranges from P to NP.

1 Introduction

The *Resource Description Framework* (RDF)[1] and its extension *RDF Schema* (RDFS)[2] are both W3C standards, and nowadays quite popular knowledge representation languages. Essentially, a statement in RDF is a triple of the form (s, p, o), allowing to state that *subject* s is related to *object* o via the *property* p. For instance,

$$(\texttt{fever}, \texttt{hasTreatment}, \texttt{paracetamol})$$

[1]http://www.w3.org/RDF/
[2]https://www.w3.org/TR/rdf-schema/

is such a triple whose intended meaning is "fever can be treated via paracetamol". RDFS is an extension of RDF providing mechanisms for describing groups of related terms and the relationships between these terms via a specific vocabulary of predicates. So, *e.g.*, the RDFS triple

$$(\texttt{paracetamol}, \mathsf{type}, \texttt{antipyretic})$$

express that "paracetamol *is an* antipyretic" (here type is the predicate for class membership specification), while

$$(\texttt{antipyretic}, \mathsf{sc}, \texttt{drug})$$

asserts that "antipyretic *is a subclass of* drug" (sc is the predicate for sub-class specification).

While both languages have been designed to represent and reason with *positive* statements only, they can not properly deal with *negative* statements such as

"opioids and antipyretics are *disjoint* classes"; (1)

"radio therapies are *non* drug treatments"; and (2)

"Ebola *has no* treatment". (3)

In particular, we may not infer that "paracetamol *is not* a treatment for Ebola". Such a statement could only be inferred with the major assumption that the Knowledge Base (KB) is complete — the so-called *Closed-World Assumption* (CWA) [27], which however is not realistic to be assumed in many cases. For instance, in medicine, it is important to distinguish between knowing about the absence of a biochemical reaction between substances, and not knowing about its existence at all, which rises then the need for explicitly stating salient *negative* statements (see, *e.g.*, [11] for a recent work about it). This is particularly true in the case in which the information about the represented world is assumed to be incomplete, — the so-called *Open World Assumption* (OWA).

Contribution. In this paper we show how to extend RDFS to express and reason with various forms of negative statements under the OWA. To do so, we start from ρdf [20], a minimal, but significant RDFS fragment that covers all essential features of RDFS, and then extend it to $\rho df_\bot^\neg$, allowing express also negative statements via a specific expressions involving negated classes/properties, disjointness relationships, and no-value existence. So, for instance, the $\rho df_\bot^\neg$ triple

$$(\texttt{opioid}, \bot_\mathsf{c}, \texttt{antipyretic})$$

expresses (1) ($\bot_\mathsf{c}$ is the vocabulary predicate for class disjointness specification),

$$(\texttt{radiotherapy}, \mathsf{sc}, \neg\texttt{drugTreatment})$$

expresses (2) (here, essentially we introduce class complements via the $\neg$ operator), while

$$(\texttt{ebola}, \neg\texttt{hasTreatment}, \star_\texttt{treatment})$$

is meant to encode (3) (here, besides property complement, we also allow the $\star_c$ operator, which is the place holder for an universally quantified variable over domain c).[3]

The main and, to the best of our knowledge, unique features of our proposal are (*cf.*, related work below): (i) $\rho df_{\bot}^{\neg}$ remains syntactically a triple language by extending ρdf with new symbols with specific semantics and there is no need to revert to the reification method to represent negative triples; (ii) the logic is defined in such a way that any RDFS reasoner/store may handle the new predicates as ordinary terms if it does not want to take account of the extra capabilities; (iii) despite negated statements, every $\rho df_{\bot}^{\neg}$ knowledge base is satisfiable, which is obtained via an intentional like four-valued semantics; (iv) the $\rho df_{\bot}^{\neg}$ entailment decision procedure is obtained from ρdf via additional inference rules favouring a potential implementation; and (v) deciding entailment in $\rho df_{\bot}^{\neg}$ ranges from P to NP.

Related Work. There have been various works in the past about extending RDFS with negative statements, or applications that would like or require to have such a feature, which we briefly summarise below and indeed inspired our work.

In [11] and related works [9, 10, 12], two types of negative statements are considered: (i) grounded negative statements of the form $\neg(s, p, o)$, with informal FOL reading $\neg p(s, o)$; and (ii) universally negative statements of the form $\neg\exists x.(s, p, x)$, meaning in FOL terms $\neg\exists x.p(s, x)$. The former type of triples have been proposed in [7] (and subsequent works, see below), while the latter has been addressed in [17]. In [11] essentially a statistical inference method is proposed to extract useful negative statements of this form, such as "Leonardo DiCaprio has never been married" and "United Kingdom is not the citizenship of Jimi Hendrix".[4] It also publishes datasets[5] that contain useful negative statements about entities in Wikidata.[6] Reasoning has not been addressed (and was not the focus of these works). Both types of negative statements are covered by $\rho df_{\bot}^{\neg}$ and, thus, our work is complementary to [11] in the sense that we describe how then to reason with such information.

In [17] the problem on how to express the non-existence of information is addressed, which has the form $No(\{(s_1, p_1, o_1), \ldots, (s_n, p_n, o_n)\})$, with informal FOL reading $\neg\exists \mathbf{x}.(p_1(s_1, o_1) \wedge \ldots \wedge p_n(s_n, o_n))$, or equivalently, $\forall \mathbf{x}.(\neg p_1(s_1, o_1) \vee \ldots \vee \neg p_n(s_n, o_n))$, where $\mathbf{x}$ are the variables occurring the triples. It shows how to represent it via the reification method and incorporate it into SPARQL[7] query answering. Reasoning is not addressed however. We consider here only the case $n = 1$ via the expression $(s, \neg p, \star_c)$ as the general case $n \geq 2$ would introduce a disjunction, which we would like to avoid for computational reasons.

In [6] and related works [8, 1, 2, 3, 4, 5, 7, 16] the authors deal with *Extended* RDF (ERDF), a non-monotonic logic, where an ERDF ontology consists of two parts: an

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[3] We refer the reader to Table 1 for an informal First-Order Logic (FOL) reading of some types of $\rho df_{\bot}^{\neg}$ triples.

[4] Optionally, triples may be annotated with a degree such as *e.g.*, "The Sultan Resort has no parking facility to degree 0.97". See *e.g.*, [33] for a general framework to deal with annotated triples.

[5] https://github.com/HibaArnaout/usefulnegations

[6] https://www.wikidata.org

[7] http://www.w3.org/TR/sparql11-query/

ERDF graph and an ERDF logic program. An ERDF graph allows *negated* RDF triples of the form $\neg(s, p, o)$, informally in FOL terms $\neg p(s, o)$, while in the body of rules all the classical connectives $\neg, \supset, \wedge, \vee, \forall, \exists$, plus the weak negation (negation-as-failure) $\sim$ are allowed. Various "stable model" semantics have been proposed. From a computational complexity point view, decision problems in ERDF are non-polynomial [8]. E.g., deciding model existence and, thus, model existence is not guaranteed, ranges from NP to PSPACE, while query answering goes from co-NP to PSPACE, depending on the setting.[8] In comparison, $\rho df_{\perp}^{\neg}$ does not use a rule layer, the triple language is more expressive, model existence is guaranteed and the computational complexity ranges between P and NP. Of course, all inference rules for $\rho df_{\perp}^{\neg}$ can be implemented in the rule layer of ERDF (and in Datalog in general).

Eventually, [15] considers $\rho df_{\perp}$ on top of which to develop a non-monotonic RDFS logic based on Rational Closure [19]. $\rho df_{\perp}$ extends ρdf allowing to express disjointness among (positive) classes and relations.

In summary, our work aims at putting all together within RDFS to deal with expressions of the form *e.g.*, (1)–(3) in a generalised way.

We proceed as follows. As next we introduce the basic notions about ρdf we will rely on. Section 3 is the main part of this paper in which we extend ρdf to $\rho df_{\perp}^{\neg}$. The paper concludes with a summary of the contributions and addresses some topics for future work.

2 Preliminaries

For the sake of our purposes, we will rely on a minimal, but significant RDFS fragment, called ρdf [20, 21], that covers the essential features of RDFS. In fact, ρdf may be considered as the logic behind RDFS and suffices to illustrate the main concepts and algorithms we will consider. ρdf is defined as the following subset of the RDFS vocabulary:

$$\rho\text{df} = \{\mathsf{sp}, \mathsf{sc}, \mathsf{type}, \mathsf{dom}, \mathsf{range}\} \ . \tag{4}$$

Informally, (i) (p, sp, q) means that property p is a *sub property* of property q; (ii) (c, sc, d) means that class c is a *sub class* of class d; (iii) (a, type, b) means that a is of *type* b; (iv) (p, dom, c) means that the *domain* of property p is c; (v) (p, range, c) means that the *range* of property p is c.

Syntax. Assume pairwise disjoint alphabets $\mathbf{U}$ (*RDF URI references*), $\mathbf{B}$ (*Blank nodes*), and $\mathbf{L}$ (*Literals*). We assume $\mathbf{U}, \mathbf{B}$, and $\mathbf{L}$ fixed, and for simplicity we will denote unions of these sets simply concatenating their names. We call elements in $\mathbf{UBL}$ *terms* (denoted $a, b, \ldots, w$), and elements in $\mathbf{B}$ *variables* (denoted x, y, z).[9] A *vocabulary* is a subset of $\mathbf{UL}$ and we assume that $\mathbf{U}$ contains the ρdf vocabulary (see Equation 4). A *triple* is of the form

$$\tau = (s, p, o) \in \mathbf{UBL} \times \mathbf{U} \times \mathbf{UBL} \ ,$$

[8] There are also many more works that use rule languages on top of RDFS, which however we are not going to discuss here (see, *e.g.*, [15].

[9] All symbols may have upper or lower script.

where $s, o \notin \rho$df. We call s the *subject*, p the *predicate*, and o the *object*. A *graph* (or *RDF Knowledge Base*) G is a set of triples τ. A subgraph is a subset of a graph. The *universe* of G, denoted $\mathtt{uni}(G)$, is the set of terms in $\mathbf{UBL}$ that occur in the triples of G. The *vocabulary* of G, denoted by $\mathtt{voc}(G)$ is the set $\mathtt{uni}(G) \cap \mathbf{UL}$. A graph is *ground* if it has no blank nodes, *i.e.*, variables. A *map* (or *variable assignment*) is as a function $\mu : \mathbf{UBL} \to \mathbf{UBL}$ preserving URIs and literals, *i.e.*, $\mu(t) = t$, for all $t \in \mathbf{UL}$. Given a graph G, we define

$$\mu(G) = \{(\mu(s), \mu(p), \mu(o)) \mid (s, p, o) \in G\} .$$

We speak of a map μ from G_1 to G_2, and write $\mu : G_1 \to G_2$, if μ is such that $\mu(G_1) \subseteq G_2$.

Example 1 (Running example)**.** *The following is a ρdf graph:*[10]

$$\begin{aligned} G = \{&(\mathtt{paracetamol}, \mathsf{type}, \mathtt{antipyretic}),\\ &(\mathtt{antipyretic}, \mathsf{sc}, \mathtt{drugTreatment}),\\ &(\mathtt{morphine}, \mathsf{type}, \mathtt{opioid}), (\mathtt{opioid}, \mathsf{sc}, \mathtt{drugTreatment}),\\ &(\mathtt{drugTreatment}, \mathsf{sc}, \mathtt{treatment}),\\ &(\mathtt{brainTumour}, \mathsf{type}, \mathtt{tumour}),\\ &(\mathtt{hasDrugTreatment}, \mathsf{sp}, \mathtt{hasTreatment}),\\ &(\mathtt{hasTreatment}, \mathsf{dom}, \mathtt{illness}),\\ &(\mathtt{hasTreatment}, \mathsf{range}, \mathtt{treatment}),\\ &(\mathtt{hasDrugTreatment}, \mathsf{range}, \mathtt{drugTreatment}),\\ &(\mathtt{fever}, \mathtt{hasDrugTreatment}, \mathtt{paracetamol})\\ &(\mathtt{brainTumour}, \mathtt{hasDrugTreatment}, \mathtt{morphine})\ \} .\end{aligned}$$

Semantics. A ρdf *interpretation* $\mathcal{I}$ over a vocabulary V is a tuple

$$\mathcal{I} = \langle \Delta_{\mathsf{R}}, \Delta_{\mathsf{P}}, \Delta_{\mathsf{C}}, \Delta_{\mathsf{L}}, \mathsf{P}[\![\cdot]\!], \mathsf{C}[\![\cdot]\!], \cdot^{\mathcal{I}} \rangle ,$$

where $\Delta_{\mathsf{R}}, \Delta_{\mathsf{P}}, \Delta_{\mathsf{C}}, \Delta_{\mathsf{L}}$ are the interpretation domains of $\mathcal{I}$, which are finite non-empty sets, and $\mathsf{P}[\![\cdot]\!], \mathsf{C}[\![\cdot]\!], \cdot^{\mathcal{I}}$ are the interpretation functions of $\mathcal{I}$. They have to satisfy:

1. Δ_{R} are the resources;
2. Δ_{P} are property names;
3. $\Delta_{\mathsf{C}} \subseteq \Delta_{\mathsf{R}}$ are the classes;
4. $\Delta_{\mathsf{L}} \subseteq \Delta_{\mathsf{R}}$ are the literal values and contains all the literals in $\mathbf{L} \cap V$;
5. $\mathsf{P}[\![\cdot]\!]$ is a function $\mathsf{P}[\![\cdot]\!] \colon \Delta_{\mathsf{P}} \to 2^{\Delta_{\mathsf{R}} \times \Delta_{\mathsf{R}}}$;

[10] For ease of presentation, we use the terms paracetomol, antipyretic, morphine and opioid to mean paracetomol-, antipyretic-, morphine- and opioid-treatement, respectively.

6. $\mathsf{C}[\![\cdot]\!]$ is a function $\mathsf{C}[\![\cdot]\!] \colon \Delta_{\mathsf{C}} \to 2^{\Delta_{\mathsf{R}}}$;

7. $\cdot^{\mathcal{I}}$ maps each $t \in \mathbf{UL} \cap V$ into a value $t^{\mathcal{I}} \in \Delta_{\mathsf{R}} \cup \Delta_{\mathsf{P}}$, where $\cdot^{\mathcal{I}}$ is the identity for literals; and

8. $\cdot^{\mathcal{I}}$ maps each variable $x \in \mathbf{B}$ into a value $x^{\mathcal{I}} \in \Delta_{\mathsf{R}}$.

As next, for space reasons and without loosing the substantial ingredients, we illustrate the so-called *reflexive-relaxed* ρdf semantics [20, Definition 12], in which the predicates sc and sp are *not* assumed to be reflexive. Informally, the notion entailment is defined using the idea of *satisfaction* of a graph under certain interpretation. Intuitively a ground triple (s, p, o) in an RDF graph G will be true under the interpretation $\mathcal{I}$ if p is interpreted as a property name, s and o are interpreted as resources, and the interpretation of the pair (s, o) belongs to the extension of the property assigned to p. Moreover, blank nodes, *i.e.*, variables, work as existential variables. Intuitively the triple (x, p, o) with $x \in \mathbf{B}$ will be true under $\mathcal{I}$ if $\mathcal{I}$ maps x into a resource s such that the pair (s, o) belongs to the extension of the property assigned to p. Formally,

Definition 1 (Model/Satisfaction/Entailment $\models$). *A ρdf interpretation $\mathcal{I}$ is a* model *of a ρdf graph G, denoted $\mathcal{I} \models G$, if and only if $\mathcal{I}$ is an interpretation over the vocabulary ρdf $\cup$ `uni`(G) such that:*

Simple:

1. *for each $(s, p, o) \in G$, $p^{\mathcal{I}} \in \Delta_{\mathsf{P}}$ and $(s^{\mathcal{I}}, o^{\mathcal{I}}) \in \mathsf{P}[\![p^{\mathcal{I}}]\!]$;*

Subproperty:

1. *$\mathsf{P}[\![\mathsf{sp}^{\mathcal{I}}]\!]$ is transitive over Δ_{P};*
2. *if $(p, q) \in \mathsf{P}[\![\mathsf{sp}^{\mathcal{I}}]\!]$ then $p, q \in \Delta_{\mathsf{P}}$ and $\mathsf{P}[\![p]\!] \subseteq \mathsf{P}[\![q]\!]$;*

Subclass:

1. *$\mathsf{P}[\![\mathsf{sc}^{\mathcal{I}}]\!]$ is transitive over Δ_{C};*
2. *if $(c, d) \in \mathsf{P}[\![\mathsf{sc}^{\mathcal{I}}]\!]$ then $c, d \in \Delta_{\mathsf{C}}$ and $\mathsf{C}[\![c]\!] \subseteq \mathsf{C}[\![d]\!]$;*

Typing I:

1. *$x \in \mathsf{C}[\![c]\!]$ if and only if $(x, c) \in \mathsf{P}[\![\mathsf{type}^{\mathcal{I}}]\!]$;*
2. *if $(p, c) \in \mathsf{P}[\![\mathsf{dom}^{\mathcal{I}}]\!]$ and $(x, y) \in \mathsf{P}[\![p]\!]$ then $x \in \mathsf{C}[\![c]\!]$;*
3. *if $(p, c) \in \mathsf{P}[\![\mathsf{range}^{\mathcal{I}}]\!]$ and $(x, y) \in \mathsf{P}[\![p]\!]$ then $y \in \mathsf{C}[\![c]\!]$;*

Typing II:

1. *for each $\mathsf{e} \in \rho df$, $\mathsf{e}^{\mathcal{I}} \in \Delta_{\mathsf{P}}$;*
2. *if $(p, c) \in \mathsf{P}[\![\mathsf{dom}^{\mathcal{I}}]\!]$ then $p \in \Delta_{\mathsf{P}}$ and $c \in \Delta_{\mathsf{C}}$;*
3. *if $(p, c) \in \mathsf{P}[\![\mathsf{range}^{\mathcal{I}}]\!]$ then $p \in \Delta_{\mathsf{P}}$ and $c \in \Delta_{\mathsf{C}}$;*
4. *if $(x, c) \in \mathsf{P}[\![\mathsf{type}^{\mathcal{I}}]\!]$ then $c \in \Delta_{\mathsf{C}}$.*

A graph G is satisfiable *if it has a model $\mathcal{I}$. Moreover, given two graphs G and H, we say that G* entails *H, denoted $G \models H$, if and only if every model of G is also a model of H.*

Example 2. *Consider Example 1. Then it can be verified that*

$$G \models \{(\texttt{fever}, \texttt{hasTreatment}, x), (x, \mathsf{type}, \texttt{drugTreatment})\} \ .$$

Deductive System for ρdf. We recap the sound and complete deductive system for ρdf [20]. In every rule, A, B, C, D, E, X and Y stand for meta-variables to be replaced by actual terms. An *instantiation* of a rule is obtained by replacing all meta-variables with terms such that all triples after the replacement are ρdf triples.

Definition 2 (Deductive rules for ρdf)**.** *The* deductive rules for ρdf *are the following:*

1. *Simple:*
   (a) $\frac{G}{G'}$ *for a map* $\mu : G' \to G$ (b) $\frac{G}{G'}$ *for* $G' \subseteq G$
2. *Subproperty:*
   (a) $\frac{(A,\mathsf{sp},B),(B,\mathsf{sp},C)}{(A,\mathsf{sp},C)}$ (b) $\frac{(D,\mathsf{sp},E),(X,D,Y)}{(X,E,Y)}$
3. *Subclass:*
   (a) $\frac{(A,\mathsf{sc},B),(B,\mathsf{sc},C)}{(A,\mathsf{sc},C)}$ (b) $\frac{(A,\mathsf{sc},B),(X,\mathsf{type},A)}{(X,\mathsf{type},B)}$
4. *Typing:*
   (a) $\frac{(D,\mathsf{dom},B),(X,D,Y)}{(X,\mathsf{type},B)}$ (b) $\frac{(D,\mathsf{range},B),(X,D,Y)}{(Y,\mathsf{type},B)}$
5. *Implicit Typing:*
   (a) $\frac{(A,\mathsf{dom},B),(D,\mathsf{sp},A),(X,D,Y)}{(X,\mathsf{type},B)}$

   (b) $\frac{(A,\mathsf{range},B),(D,\mathsf{sp},A),(X,D,Y)}{(Y,\mathsf{type},B)}$

Definition 3 (Derivation $\vdash$)**.** *Let G and H be ρdf-graphs. $G \vdash H$ if and only if there exists a sequence of graphs $P_1, P_2, \ldots, P_k$ with $P_1 = G$ and $P_k = H$ and for each j ($2 \leq j \leq k$) one of the following cases hold:*

- *there is a map $\mu : P_j \to P_{j-1}$ (rule (1a));*
- *$P_j \subseteq P_{j-1}$ (rule (1b));*
- *there is an instantiation R/R' of one of the rules (2)-(5), such that $R \subseteq P_{j-1}$ and $P_j = P_{j-1} \cup R'$.*

Such a sequence of graphs is called a proof of $G \vdash H$. Whenever $G \vdash H$, we say that the graph H is derived from the graph G. Each pair (P_{j-1}, P_j), $1 \leq j \leq k$ is called a step of the proof which is labeled by the respective instantiation R/R' of the rule applied at the step.

Please note that if $G \vdash H$ then H is indeed a graph. Finally, the *closure* of a graph G, denoted $\mathsf{Cl}(G)$, is defined as

$$\mathsf{Cl}(G) = \{\tau \mid G \vdash^* \tau\} ,$$

where $\vdash^*$ is as $\vdash$ except that rule $(1a)$ is excluded.

Example 3. *Consider Example 1. Then it can be verified that*

$$\mathsf{Cl}(G) \supseteq \{(\texttt{morphine}, \mathsf{type}, \texttt{drugTreatment}), (\texttt{brainTumour}, \mathsf{type}, \texttt{illness})\} .$$

The following proposition recaps salient results taken from [18, 20, 32]

Proposition 1. *Every ρdf-graph is satisfiable. Moreover, let G and H be ρdf-graphs. Then*

1. *$G \vdash H$ if and only if $G \models H$;*
2. *if $G \models H$ then there is a proof of H from G where rule $(1a)$ is used at most once and at the end;*
3. *the closure of G is unique and $|\mathsf{Cl}(G)| \in \Theta(|G|^2)$;*
4. *deciding $G \models H$ is an NP-complete problem;*
5. *if G is ground then $\mathsf{Cl}(G)$ can be determined without using implicit typing rules (5);*
6. *if H is ground, then $G \models H$ if and only if $H \subseteq \mathsf{Cl}(G)$;*
7. *There is no triple τ such that $\emptyset \models \tau$.*

Remark 1. *Please note that: (i) a proof of NP-completeness of point 4. above can be found in [32, Proposition 2.19] via a reduction of the k-clique problem encoding an undirected graph G into a ρdf graph G' (an edge $\langle v, w\rangle$ is encoded via two triples (v, e, w) and (w, e, v)) and H' consists of the triples (x, e, y), where x and y are distinct variables of new set of k blank nodes. Then, G has a clique of size $\geq k$ iff $G' \models H'$, i.e., there is a map $\mu\colon H' \to G'$; and (ii) concerning the size of the closure, the lower bound is determined by triples $(p_1, \mathsf{sp}, p_2), \ldots, (p_n, \mathsf{sp}, p_{n+1})$, whose closure's size is $\Omega(n^2)$ via rule $(2a)$. The upper bound follows by an analysis of the rules, where the important point is the propagation of triples (s, p, o) via rule $(2b)$. This gives at most a quadratic upper bound (for triples with fixed predicate, the quadratic bound is trivial).*

Please note that from Proposition 1 it follows that deciding if, for two ground ρdf-graphs G and H, $G \models H$ can be done in time $O(|H||G|^2)$ by computing first the closure of G and then check whether H is in that closure. However, [20] presents also an alternative method not requiring to compute the closure with a computational benefit as illustrated by the following proposition:

Proposition 2. *Let G and H be two ground ρdf graphs. Then deciding if $G \models H$ can be done in time $O(|H||G| \log |G|)$. The result hods also in case each triple in H has at most one blank node.*

3 Extending ρdf with Negative Statements

In this section, we show how to extend ρdf allowing to represent negative statements.

3.1 Syntax

To start with, consider a new pair of predicates, $\bot_{\mathsf{c}}$ and $\bot_{\mathsf{p}}$, representing disjoint information: *e.g.*, (i) $(c, \bot_{\mathsf{c}}, d)$ indicates that the classes c and d are disjoint; analogously, (ii) $(p, \bot_{\mathsf{p}}, q)$ indicates that the properties p and q are disjoint.

We call $\rho df_\bot$ the vocabulary obtained from ρdf by adding $\bot_{\mathsf{c}}$ and $\bot_{\mathsf{p}}$, that is,

$$\rho df_\bot = \rho \text{df} \cup \{\bot_{\mathsf{c}}, \bot_{\mathsf{p}}\} \ . \tag{5}$$

Like for ρdf, we assume that $\mathbf{U}$ contains the $\rho df_\bot$ vocabulary. Now we extend the alphabet $\mathbf{U}$ in the following way:

1. for each (*atomic*) resource $r \in \mathbf{U} \setminus \rho df_\bot$, we add to $\mathbf{U}$ a new *negated resource* $\neg r$ of r. Let $\mathbf{U}'$ be the resulting alphabet. We will use the convention that $\neg\neg r$ is r. Informally, $\neg r$ is intended to represent the complement of r. So, for instance, $(\mathtt{paracetamol}, \mathtt{type}, \neg\mathtt{opioid})$ may encode "paracetamol is a non opioid treatmen)";
2. for each resource $c \in \mathbf{U}' \setminus \rho df_\bot$, we add to $\mathbf{U}'$ a new resource of the form $\star_c$. Let $\mathbf{U}''$ be the resulting alphabet. Informally, *e.g.*, a triple $(s, p, \star_c)$ represents an universal quantification on the third argument over instance of class c, *i.e.*, (s, p, t) is true for all $t \in \mathbf{UL}$ that are instances of the class c. For instance, $(\mathtt{ebola}, \neg\mathtt{hasTreatment}, \star_{\mathtt{treatment}})$ may encode (3);[11]
3. finally, let $\mathbf{U}$ be $\mathbf{U}''$.

Now, the definition of $\rho df_\bot^\neg$*-triples* extends the one for ρdf-triples in the following following way:

Definition 4 ($\rho df_\bot^\neg$-triple)**.** *A* $\rho df_\bot^\neg$-triple *is of the form*

$$\tau = (s, p, o) \in \mathbf{UBL} \times \mathbf{U} \times \mathbf{UBL} \ ,$$

where

1. $s, o \notin \rho df_\bot$*;*
2. p *is not of the form* $\star_c$*;*
3. s *and* o *can not be both of the form* $\star_c$*;*
4. *if* $p \in \rho df_\bot$ *then neither* s *nor* o *are of the form* $\star_c$*.*

In Table 1, to ease the reading, we provide an informal FOL reading of various additional (non exhaustive) types of triples supported in $\rho df_\bot^\neg$.

[11] In the sense that "none of the treatments are treatments for ebola"

$\rho df_\bot^\neg$	FOL
$(s, \neg p, o)$	$\neg p(s, o)$
$(s, \neg p, x)$	$\exists x.\neg p(s, x)$
$(a, \mathsf{type}, \neg c)$	$\neg c(a)$
$(c, \mathsf{sc}, \neg d)$	$\forall x.c(x) \to \neg d(x)$
$(p, \mathsf{dom}, \neg c)$	$\forall x \forall y.p(x, y) \to \neg c(x)$
$(\neg p, \mathsf{range}, d)$	$\forall x \forall y.\neg p(x, y) \to d(y)$
$(c, \bot_\mathsf{c}, \neg d)$	$\forall x.c(x) \land \neg d(x) \to \bot$
$(\neg p, \bot_\mathsf{p}, q)$	$\forall x \forall y.\neg p(x, y) \land q(x, y) \to \bot$
$(\star_c, p, o)$	$\forall x.c(x) \to p(x, o)$
$(s, \neg p, \star_c)$	$\forall y.c(y) \to \neg p(s, y)$ (*i.e.*, $\neg \exists y.c(y) \land p(s, y)$)

Table 1: Informal FOL reading of some types of $\rho df_\bot^\neg$-triples.

Example 4. *In the context of Example 1, let us extend the graph G with:*

$$
\begin{aligned}
G \coloneqq G \cup \{ & (\texttt{opioid}, \bot_\mathsf{c}, \texttt{antipyretic}), \\
& (\neg\texttt{drugTreatment}, \mathsf{sc}, \texttt{treatment}), \\
& (\neg\texttt{hasDrugTreatment}, \mathsf{sp}, \texttt{hasTreatment}), \\
& (\neg\texttt{hasDrugTreatment}, \mathsf{range}, \neg\texttt{drugTreatment}), \\
& (\texttt{brainTumour}, \neg\texttt{hasDrugTreatment}, \texttt{radioTherapy}), \\
& (\neg\texttt{hasTreatment}, \mathsf{dom}, \texttt{illness}), \\
& (\neg\texttt{hasTreatment}, \mathsf{range}, \texttt{treatment}), \\
& (\texttt{ebola}, \neg\texttt{hasTreatment}, \star_{\texttt{treatment}}) \} \, .
\end{aligned}
$$

3.2 Semantics

The semantics of $\rho df_\bot^\neg$ has the following objectives:

1. we are going to accommodate the new constructs in such a way that the resulting deductive system will be as for ρdf, plus some additional rules. In this way, any RDFS reasoner/store may handle the new triples as ordinary triples if it does not want to take account of the extra inference capabilities;

2. the semantics has to be such that, despite introducing negative statements, all $\rho df_\bot^\neg$ graphs have a canonical model (see Corollary 1 later on), and, thus, $\rho df_\bot^\neg$ remains a *paraconsistent* logic; and

3. deciding entailment in $\rho df_\bot^\neg$ still ranges from P to NP.

To do so, we will consider a *four-valued* logic semantics [14] variant of the semantics for ρdf. Specifically, we will have *positive extensions* of $\mathsf{P}[\![]\!]$ and $\mathsf{C}[\![]\!]$ (denoted $\mathsf{P}^+[\![]\!]$ and $\mathsf{C}^+[\![]\!]$, respectively) and *negative extensions* of $\mathsf{P}[\![]\!]$ and $\mathsf{C}[\![]\!]$ (denoted $\mathsf{P}^-[\![]\!]$ and $\mathsf{C}^-[\![]\!]$, respectively). Roughly, $\mathsf{C}^+[\![c]\!]$ will denote the set of resources known *to be* instances of class c, while $\mathsf{C}^-[\![c]\!]$ will denote the set of resources known *not to be* instances of class c (for properties the case is similar). Note that positive and negative extensions need not to be the complement of each other: *e.g.*, $r \notin \mathsf{C}^+[\![c]\!]$ does not imply necessarily that $r \in \mathsf{C}^-[\![c]\!]$ as $\mathsf{C}^-[\![c]\!]$ will not enforced to be *e.g.*, $\Delta_\mathsf{R} \setminus \mathsf{C}^+[\![c]\!]$.

The idea of having separate positive and negative extensions is not new at all and we may find already traces of it back in the mid 80s with the seminal work of Patel-Schneider [22, 23, 24, 25, 26] in which four-valued variants of *Terminological Logics* (TLs), *viz.*, the so-called *Description Logics* (DLs) [13] nowadays, have been proposed with the aim to obtain some gain from a computational complexity point of view. Later the works [28, 29, 30, 31] have been inspired by the same idea, though also to model some sort of *relevance entailment*, besides being paraconsistent. More recently, a similar idea has been considered also in the context of RDFS [7, 5, 4, 8], which is also the semantics we start from and are going to adapt and extend to meet the before mentioned objectives.

A $\rho df_\bot^\neg$ *interpretation* $\mathcal{I}$ over a vocabulary V is now a tuple

$$\mathcal{I} = \langle \Delta_\mathsf{R}, \Delta_\mathsf{P}, \Delta_\mathsf{C}, \Delta_\mathsf{L}, \mathsf{P}^+[\![\cdot]\!], \mathsf{P}^-[\![\cdot]\!], \mathsf{C}^+[\![\cdot]\!], \mathsf{C}^-[\![\cdot]\!], \cdot^\mathcal{I} \rangle \ ,$$

where $\Delta_\mathsf{R}, \Delta_\mathsf{P}, \Delta_\mathsf{C}, \Delta_\mathsf{L}$ are the interpretation domains of $\mathcal{I}$, which are finite non-empty sets, and $\mathsf{P}^+[\![\cdot]\!], \mathsf{P}^-[\![\cdot]\!], \mathsf{C}^+[\![\cdot]\!], \mathsf{C}^-[\![\cdot]\!], \cdot^\mathcal{I}$ are the interpretation functions of $\mathcal{I}$. They have to satisfy:

1. Δ_R are the resources;
2. Δ_P are property names;
3. $\Delta_\mathsf{C} \subseteq \Delta_\mathsf{R}$ are the classes;
4. for each domain $\Delta_\mathsf{R}, \Delta_\mathsf{P}$ and Δ_C, for each term t in it, there is an unique designated complement term of t, denoted $\neg t$, in it;[12]
5. $\Delta_\mathsf{L} \subseteq \Delta_\mathsf{R}$ are the literal values and contains all the literals in $\mathbf{L} \cap V$;
6. $\mathsf{P}^+[\![\cdot]\!]$ and $\mathsf{P}^-[\![\cdot]\!]$ are functions $\Delta_\mathsf{P} \to 2^{\Delta_\mathsf{R} \times \Delta_\mathsf{R}}$ such that $\mathsf{P}^+[\![\neg p]\!] = \mathsf{P}^-[\![p]\!]$, for each $p \in \Delta_\mathsf{P}$;
7. $\mathsf{C}^+[\![\cdot]\!]$ and $\mathsf{C}^-[\![\cdot]\!]$ are functions $\Delta_\mathsf{C} \to 2^{\Delta_\mathsf{R}}$ such that $\mathsf{C}^+[\![\neg c]\!] = \mathsf{C}^-[\![c]\!]$, for each $c \in \Delta_\mathsf{C}$;
8. $\cdot^\mathcal{I}$ maps each $t \in \mathbf{UL} \cap V$, that is not of the form $\star_c$, into a value $t^\mathcal{I} \in \Delta_\mathsf{R} \cup \Delta_\mathsf{P}$, such that $(\neg t)^\mathcal{I} = \neg\, t^\mathcal{I}$ and $\cdot^\mathcal{I}$ is the identity for literals; and
9. $\cdot^\mathcal{I}$ maps each variable $x \in \mathbf{B}$ into a value $x^\mathcal{I} \in \Delta_\mathsf{R}$.

In the following, we define

$$\begin{array}{rcl}
\mathsf{P}^+[\![p]\!]\!\uparrow & = & \{x \in \Delta_\mathsf{R} \mid (x,y) \in \mathsf{P}^+[\![p]\!]\} \\
\mathsf{P}^+[\![p]\!]\!\downarrow & = & \{y \in \Delta_\mathsf{R} \mid (x,y) \in \mathsf{P}^+[\![p]\!]\} \\
\mathsf{P}^-[\![p]\!]\!\uparrow & = & \{x \in \Delta_\mathsf{R} \mid (x,y) \in \mathsf{P}^-[\![p]\!]\} \\
\mathsf{P}^-[\![p]\!]\!\downarrow & = & \{y \in \Delta_\mathsf{R} \mid (x,y) \in \mathsf{P}^-[\![p]\!]\}
\end{array}$$

[12] As for $\mathbf{U}$, we will use the convention that $\neg\neg t$ is t.

as the projections of the property extension functions $\mathsf{P}^+[\![]\!]$ and $\mathsf{P}^-[\![]\!]$ on the first and second argument, respectively.

Now, the model/satisfaction/entailment definitions for ρdf are generalised to $\rho df_\perp^\neg$ as follows:

Definition 5 (Model/Satisfaction/Entailment $\models_{\neg,}^{\perp}$). *A $\rho df_\perp^\neg$ interpretation $\mathcal{I}$ is a $\rho df_\perp^\neg$-*model *of a $\rho df_\perp^\neg$ graph G, denoted $\mathcal{I} \models_{\neg,}^{\perp} G$, if and only if $\mathcal{I}$ is a $\rho df_\perp^\neg$-interpretation over the vocabulary $\rho df_\perp^\neg \cup \texttt{uni}(G)$ such that:*

Simple:

1. *if $(s,p,o) \in G$ and neither s nor o are of the form $\star_c$, then $p^\mathcal{I} \in \Delta_\mathsf{P}$ and $(s^\mathcal{I}, o^\mathcal{I}) \in \mathsf{P}^+[\![p^\mathcal{I}]\!]$;*
2. *if $(s,p,\star_c) \in G$, then $p^\mathcal{I} \in \Delta_\mathsf{P}, c^\mathcal{I} \in \Delta_\mathsf{C}$ and $(s^\mathcal{I}, y) \in \mathsf{P}^+[\![p^\mathcal{I}]\!]$, for all $y \in \mathsf{C}^+[\![c^\mathcal{I}]\!]$;*
3. *if $(\star_c,p,s) \in G$, then $p^\mathcal{I} \in \Delta_\mathsf{P}, c^\mathcal{I} \in \Delta_\mathsf{C}$ and $(x, s^\mathcal{I}) \in \mathsf{P}^+[\![p^\mathcal{I}]\!]$, for all $x \in \mathsf{C}^+[\![c^\mathcal{I}]\!]$;*
4. *if $(s,p,\star_c) \in G$, then $p^\mathcal{I} \in \Delta_\mathsf{P}, c^\mathcal{I} \in \Delta_\mathsf{C}$ and $y \in \mathsf{C}^-[\![c^\mathcal{I}]\!]$, for all $(s^\mathcal{I}, y) \in \mathsf{P}^-[\![p^\mathcal{I}]\!]$;*
5. *if $(\star_c,p,s) \in G$, then $p^\mathcal{I} \in \Delta_\mathsf{P}, c^\mathcal{I} \in \Delta_\mathsf{C}$ and $x \in \mathsf{C}^-[\![c^\mathcal{I}]\!]$, for all $(x, s^\mathcal{I}) \in \mathsf{P}^-[\![p^\mathcal{I}]\!]$;*

Subproperty:

1. *$\mathsf{P}^+[\![\mathsf{sp}^\mathcal{I}]\!]$ is transitive over Δ_P;*
2. *if $(p,q) \in \mathsf{P}^+[\![\mathsf{sp}^\mathcal{I}]\!]$ then $p,q \in \Delta_\mathsf{P}$ and $\mathsf{P}^+[\![p]\!] \subseteq \mathsf{P}^+[\![q]\!]$;*
3. *$(p,q) \in \mathsf{P}^+[\![\mathsf{sp}^\mathcal{I}]\!]$ if and only if $(\neg q, \neg p) \in \mathsf{P}^+[\![\mathsf{sp}^\mathcal{I}]\!]$;*

Subclass:

1. *$\mathsf{P}^+[\![\mathsf{sc}^\mathcal{I}]\!]$ is transitive over Δ_C;*
2. *if $(c,d) \in \mathsf{P}^+[\![\mathsf{sc}^\mathcal{I}]\!]$ then $c,d \in \Delta_\mathsf{C}$ and $\mathsf{C}^+[\![c]\!] \subseteq \mathsf{C}^+[\![d]\!]$;*
3. *$(c,d) \in \mathsf{P}^+[\![\mathsf{sc}^\mathcal{I}]\!]$ if and only if $(\neg d, \neg c) \in \mathsf{P}^+[\![\mathsf{sc}^\mathcal{I}]\!]$;*

Typing I:

1. *$x \in \mathsf{C}^+[\![c]\!]$ if and only if $(x,c) \in \mathsf{P}^+[\![\mathsf{type}^\mathcal{I}]\!]$;*
2. *if $(p,c) \in \mathsf{P}^+[\![\mathsf{dom}^\mathcal{I}]\!]$ and $(x,y) \in \mathsf{P}^+[\![p]\!]$ then $x \in \mathsf{C}^+[\![c]\!]$;*
3. *if $(p,c) \in \mathsf{P}^+[\![\mathsf{range}^\mathcal{I}]\!]$ and $(x,y) \in \mathsf{P}^+[\![p]\!]$ then $y \in \mathsf{C}^+[\![c]\!]$;*
4. *if $(p,c) \in \mathsf{P}^+[\![\mathsf{dom}^\mathcal{I}]\!]$, $x \in \mathsf{C}^-[\![c]\!]$ and $y \in \mathsf{P}^+[\![p]\!]\!\downarrow$ then $(x,y) \in \mathsf{P}^-[\![p]\!]$;*
5. *if $(p,c) \in \mathsf{P}^+[\![\mathsf{range}^\mathcal{I}]\!]$, $y \in \mathsf{C}^-[\![c]\!]$ and $x \in \mathsf{P}^+[\![p]\!]\!\uparrow$ then $(x,y) \in \mathsf{P}^-[\![p]\!]$;*

Typing II:

1. *For each $\mathsf{e} \in \rho df_\perp^\neg$, $\mathsf{e}^\mathcal{I} \in \Delta_\mathsf{P}$;*

2. *if* $(p, c) \in \mathsf{P}^+[\![\mathsf{dom}^{\mathcal{I}}]\!]$ *then* $p \in \Delta_{\mathsf{P}}$ *and* $c \in \Delta_{\mathsf{C}}$*;*
3. *if* $(p, c) \in \mathsf{P}^+[\![\mathsf{range}^{\mathcal{I}}]\!]$ *then* $p \in \Delta_{\mathsf{P}}$ *and* $c \in \Delta_{\mathsf{C}}$*;*
4. *if* $(x, c) \in \mathsf{P}^+[\![\mathsf{type}^{\mathcal{I}}]\!]$ *then* $c \in \Delta_{\mathsf{C}}$*;*

Disjointness I:

1. *if* $(c, d) \in \mathsf{P}^+[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$ *then* $c, d \in \Delta_{\mathsf{C}}$*;*
2. *if* $(p, q) \in \mathsf{P}^+[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$ *then* $p, q \in \Delta_{\mathsf{P}}$*;*
3. $\mathsf{P}^+[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$ *is symmetric, sub-transitive and exhaustive over* Δ_{C}*;*
   Symmetry: *if* $(c, d) \in \mathsf{P}^+[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$*, then* $(d, c) \in \mathsf{P}^+[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$*;*
   Sub-Transitivity: *if* $(c, d) \in \mathsf{P}^+[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$ *and* $(e, c) \in \mathsf{P}^+[\![\mathsf{sc}^{\mathcal{I}}]\!]$*, then* $(e, d) \in \mathsf{P}^+[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$*;*
   Exhaustive: *if* $(c, c) \in \mathsf{P}^+[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$ *and* $d \in \Delta_{\mathsf{C}}$ *then* $(c, d) \in \mathsf{P}^+[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$*;*
4. $\mathsf{P}^+[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$ *is symmetric, sub-transitive and exhaustive over* Δ_{P}*;*
   Symmetry: *If* $(p, q) \in \mathsf{P}^+[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$*, then* $(q, p) \in \mathsf{P}^+[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$*;*
   Sub-Transitivity: *if* $(p, q) \in \mathsf{P}^+[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$ *and* $(r, p) \in \mathsf{P}^+[\![\mathsf{sp}^{\mathcal{I}}]\!]$*, then* $(r, q) \in \mathsf{P}^+[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$*;*
   Exhaustive: *if* $(p, p) \in \mathsf{P}^+[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$ *and* $q \in \Delta_{\mathsf{P}}$ *then* $(p, q) \in \mathsf{P}^+[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$*;*

Disjointness II:

1. *if* $(p, c) \in \mathsf{P}^+[\![\mathsf{dom}^{\mathcal{I}}]\!]$*,* $(q, d) \in \mathsf{P}^+[\![\mathsf{dom}^{\mathcal{I}}]\!]$*, and* $(c, d) \in \mathsf{P}^+[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$*, then* $(p, q) \in \mathsf{P}^+[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$*;*
2. *if* $(p, c) \in \mathsf{P}^+[\![\mathsf{range}^{\mathcal{I}}]\!]$*,* $(q, d) \in \mathsf{P}^+[\![\mathsf{range}^{\mathcal{I}}]\!]$*, and* $(c, d) \in \mathsf{P}^+[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$*, then* $(p, q) \in \mathsf{P}^+[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$*;*
3. $(c, d) \in \mathsf{P}^+[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$ *if and only if* $(c, \neg d) \in \mathsf{P}^+[\![\mathsf{sc}^{\mathcal{I}}]\!]$*;*
4. $(p, q) \in \mathsf{P}^+[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$ *if and only if* $(p, \neg q) \in \mathsf{P}^+[\![\mathsf{sp}^{\mathcal{I}}]\!]$.

A graph G *is* $\rho df_{\bot}^{\neg}$-satisfiable *if it has a* $\rho df_{\bot}^{\neg}$*-model* $\mathcal{I}$*. Moreover, given two* $\rho df_{\bot}^{\neg}$*-graphs* G *and* H*, we say that* G $\rho df_{\bot}^{\neg}$-entails H*, denoted* $G \models^{\bot}_{\neg} H$*, if and only if every* $\rho df_{\bot}^{\neg}$*-model of* G *is also a* $\rho df_{\bot}^{\neg}$*-model of* H*.*

In the following, if clear from the context, for ease of presentation, we will omit the prefix $\rho df_{\bot}^{\neg}$-.

Remark 2 (About the semantics)**.** *Concerning Definition 5, let us note the following:*

1. *the positive extension of a negated class is the negative extension of that class;*
2. *by construction, we also have that for* $(s, \neg p, o) \in G$*,* $(s^{\mathcal{I}}, o^{\mathcal{I}}) \in \mathsf{P}^+[\![(\neg p)^{\mathcal{I}}]\!] = \mathsf{P}^+[\![\neg\, p^{\mathcal{I}}]\!] = \mathsf{P}^-[\![p^{\mathcal{I}}]\!]$*. That is,* $(s, \neg p, o) \in G$ *states that "*(s, o) *belongs to the negative extension of* p*, i.e.,* s *has a non* p *that is an* o*";*
3. *by construction, e.g., if* $(c, d) \in \mathsf{P}^+[\![\mathsf{sc}^{\mathcal{I}}]\!]$ *then also* $\mathsf{C}^-[\![d]\!] \subseteq \mathsf{C}^-[\![c]\!]$*;*

4. *by construction,* $x \in \mathsf{C}^-[\![c]\!]$ *if and only if* $(x, \neg c) \in \mathsf{P}^+[\![\mathsf{type}^{\mathcal{I}}]\!]$*;*

5. *concerning e.g., point 5 in Typing I, from a FOL perspective, the reading of* (p, range, c) *is* $\forall x.\forall y.p(x, y) \to c(y)$ *and* $\forall x.\forall y \in p{\downarrow} .\neg c(y) \to \neg p(x, y)$*. In the latter case, the aim is to limit the universal quantification in a reasonable way.*

6. *the presence of e.g.,* $(a, \mathsf{type}, b), (a, \mathsf{type}, c)$ *and* $(b, \perp_{\mathsf{c}}, c)$ *in a graph does not preclude its satisfiability. In fact, a* $\rho df_\perp^\neg$ *graph will always be satisfiable (see Corollary 1 later on) avoiding, thus, the* ex falso quodlibet *principle. This is in line with the* ρdf *semantics [20].*

Example 5. *Consider Example 4. Then, it may be verified that*

$$G \models^{\perp}_{\neg} \{(\texttt{brainTumor}, \texttt{hasTreatment}, x), (x, \mathsf{type}, \neg\texttt{antipyretic})\} \quad (6)$$

$$G \models^{\perp}_{\neg} (\texttt{ebola}, \neg\texttt{hasTreatment}, \texttt{paracetomol}) \quad (7)$$

$$G \not\models^{\perp}_{\neg} (\texttt{ebola}, \neg\texttt{hasTreatment}, \texttt{ebola})\ . \quad (8)$$

Note that the last entailment does not hold as "ebola is not a treatment", which instead would hold without the restriction on the domain of the universal quantification.

Remark 3. *As anticipated in the related work section, [17] considers expressions of the form*

$$No(\{(s_1, p_1, o_1), \ldots, (s_n, p_n, o_n)) \quad (9)$$

in informal FOL terms $\neg\exists\mathbf{x}.(p_1(s_1, o_1) \wedge \ldots \wedge p_n(s_n, o_n))$*, which is the same as,* $\forall\mathbf{x}.(\neg p_1(s_1, o_1) \vee \ldots \vee \neg p_n(s_n, o_n))$*, where* $\mathbf{x}$ *are the variables occurring the triples. For instance,*

$$No(\{(\texttt{obama}, \texttt{child}, x), (x, \texttt{gender}, \texttt{male})\}) \quad (10)$$

expresses that "Obama has no son".

$\rho df_\perp^\neg$ *considers only the case* $n = 1$ *in* (9) *via the expression* $(s, \neg p, \star_c)$ *as the general case would introduce a disjunction, which we would like to avoid for computational reasons. Nevertheless, we may consider the option to use e.g.,*

$$(\texttt{obama}, \neg\texttt{child}, \star_{\texttt{malePerson}})$$

instead to express (10).

3.3 Deductive System for $\rho df_\perp^\neg$

We now present a deductive system for $\rho df_\perp^\neg$.

Definition 6 (Deductive rules for $\rho df_\perp^\neg$). *The* deductive rules for $\rho df_\perp^\neg$ *are all rules for* ρdf *to which we add the following rules (*Z *is an additional meta-variable):*

2. *Subproperty:*

   $(c)\quad \frac{(A,\mathsf{sp},B)}{(\neg B,\mathsf{sp},\neg A)}$

   $(d)\quad \frac{(A,D,\star_C),(D,\mathsf{sp},E)}{(A,E,\star_C)} \qquad (e)\quad \frac{(\star_C,D,A),(D,\mathsf{sp},E)}{(\star_C,E,A)}$

3. *Subclass:*

   $(c)\quad \frac{(A,\mathsf{sc},B)}{(\neg B,\mathsf{sc},\neg A)}$

   $(d)\quad \frac{(A,D,\star_C),(B,\mathsf{sc},C)}{(A,D,\star_B)} \qquad (e)\quad \frac{(\star_C,D,A),(B,\mathsf{sc},C)}{(\star_B,D,A)}$

4. *Typing:*

   $(c)\quad \frac{(D,\mathsf{dom},B),(X,\mathsf{type},\neg B),(Z,D,Y)}{(X,\neg D,Y)}$

   $(d)\quad \frac{(D,\mathsf{range},B),(Y,\mathsf{type},\neg B),(X,D,Z)}{(X,\neg D,Y)}$

   $(e)\quad \frac{(A,D,\star_C),(Y,\mathsf{type},C)}{(A,D,Y)} \qquad (f)\quad \frac{(\star_C,D,B),(X,\mathsf{type},C)}{(X,D,B)}$

   $(g)\quad \frac{(A,D,\star_C),(A,\neg D,Y)}{(Y,\mathsf{type},\neg C)} \qquad (h)\quad \frac{(\star_C,D,B),(X,\neg D,B)}{(X,\mathsf{type},\neg C)}$

6. *Conceptual Disjointness:*

   $(a)\quad \frac{(A,\bot_\mathsf{c},B)}{(B,\bot_\mathsf{c},A)} \qquad (b)\quad \frac{(A,\bot_\mathsf{c},B),(C,\mathsf{sc},A)}{(C,\bot_\mathsf{c},B)}$

   $(c)\quad \frac{(A,\bot_\mathsf{c},A)}{(A,\bot_\mathsf{c},B)} \qquad (d)\quad \frac{(A,\bot_\mathsf{c},B)}{(A,\mathsf{sc},\neg B)}$

   $(e)\quad \frac{(A,\mathsf{sc},B)}{(A,\bot_\mathsf{c},\neg B)}$

7. *Predicate Disjointness:*

   $(a)\quad \frac{(A,\bot_\mathsf{p},B)}{(B,\bot_\mathsf{p},A)} \qquad (b)\quad \frac{(A,\bot_\mathsf{p},B),(C,\mathsf{sp},A)}{(C,\bot_\mathsf{p},B)}$

   $(c)\quad \frac{(A,\bot_\mathsf{p},A)}{(A,\bot_\mathsf{p},B)} \qquad (d)\quad \frac{(A,\bot_\mathsf{p},B)}{(A,\mathsf{sp},\neg B)}$

   $(e)\quad \frac{(A,\mathsf{sp},B)}{(A,\bot_\mathsf{p},\neg B)}$

8. *Crossed Disjointness:*

   $(a)\quad \frac{(A,\mathsf{dom},C),(B,\mathsf{dom},D),(C,\bot_\mathsf{c},D)}{(A,\bot_\mathsf{p},B)}$

   $(b)\quad \frac{(A,\mathsf{range},C),(B,\mathsf{range},D),(C,\bot_\mathsf{c},D)}{(A,\bot_\mathsf{p},B)}$

Now, the definition of derivation among $\rho df_\bot^\neg$-graphs G and H, denoted $G \vdash^{\bot}_{\neg} H$, is as for ρdf (see Definition 3), except that, of course, we now consider all rules of Definition 6 instead. Similarly, the *$\rho df_\bot^\neg$-closure* of a graph G, denoted $\mathsf{Cl}_\bot^\neg(G)$, is defined as

$$\mathsf{Cl}_\bot^\neg(G) = \{\tau \mid G \vdash^{\bot}_{\neg}{}^{*}\, \tau\}\ ,$$

where $\models^{\perp}_{\neg}{}^{*}$ is as $\models^{\perp}_{\neg}$ except that rule $(1a)$ is excluded.

Example 6. *The following is a simple proof a* (6)*:*

(1)	$(\texttt{opioid}, \perp_{\mathsf{c}}, \texttt{antipyretic})$	*Rule* $(1b)$
(2)	$(\texttt{opioid}, \mathsf{sc}, \neg\texttt{antipyretic})$	*Rule* $(6d) : (1)$
(3)	$(\texttt{morphine}, \mathsf{type}, \texttt{opioid})$	*Rule* $(1b)$
(4)	$(\texttt{morphine}, \mathsf{type}, \neg\texttt{antipyretic})$	*Rule* $(3b) : (2), (3)$
(5)	$(\texttt{brainTumour}, \texttt{hasDrugTreatment}, \texttt{morphine})$	*Rule* $(1b)$
(10)	$(\texttt{brainTumor}, \texttt{hasTreatment}, x)$ $(x, \mathsf{type}, \neg\texttt{antipyretic})$	 *Rule* $(1a)$*:* $(4), (5)$

In the following, we will also assume that the definition of entailment $\models$ is extended naturally to $\rho df_{\perp}^{\neg}$-graphs by considering $\perp_{\mathsf{c}}, \perp_{\mathsf{p}}, \neg p$ and $\star_c$ as resources without any specific semantic constraint. In a similar way, we assume $\vdash$ (and $\mathsf{Cl}(\cdot)$) to be extended to $\rho df_{\perp}^{\neg}$-graphs by assuming that triples involving $\perp_{\mathsf{c}}, \perp_{\mathsf{p}}, \neg p$ and $\star_c$ are considered as ρdf triples. Then, the following can easily be proven:

Proposition 3. *Let G and H be two $\rho df_{\perp}^{\neg}$-graphs. Then,*

1. *if $G \models H$ then $G \models^{\perp}_{\neg} H$;*
2. *if $G \vdash H$ then $G \vdash^{\perp}_{\neg} H$;*
3. $\mathsf{Cl}(G) \subseteq \mathsf{Cl}_{\perp}^{\neg}(G)$.

Of course, conditions 1.-3. above do not hold in general for the opposite direction. For instance, for $G = \{(a, \perp_{\mathsf{c}}, b)\}$ we have $G \not\models (a, \mathsf{sc}, \neg b)$, $G \not\vdash (a, \mathsf{sc}, \neg b)$ and $(a, \mathsf{sc}, \neg b) \notin \mathsf{Cl}(G)$, but $G \models^{\perp}_{\neg} (a, \mathsf{sc}, \neg b)$ and $(a, \mathsf{sc}, \neg b) \in \mathsf{Cl}_{\perp}^{\neg}(G)$.

The following proposition defines the construction of the *canonical model* for $\rho df_{\perp}^{\neg}$ graphs and extends the result for ρdf (see Proposition 1), showing then that all $\rho df_{\perp}^{\neg}$-graphs G are satisfiable.

Proposition 4 ($\rho df_{\perp}^{\neg}$ Canonical model)**.** *Given a $\rho df_{\perp}^{\neg}$-graph G, define a $\rho df_{\perp}^{\neg}$ interpretation $\mathcal{I}_G$ as a tuple*

$$\mathcal{I}_G = \langle \Delta_{\mathsf{R}}, \Delta_{\mathsf{P}}, \Delta_{\mathsf{C}}, \Delta_{\mathsf{L}}, \mathsf{P}^{+}[\![\cdot]\!], \mathsf{P}^{-}[\![\cdot]\!], \mathsf{C}^{+}[\![\cdot]\!], \mathsf{C}^{-}[\![\cdot]\!], \cdot^{\mathcal{I}_G} \rangle$$

such that:

1. $\Delta_{\mathsf{R}} \coloneqq \texttt{uni}(G) \cup \{\neg r \mid r \in \texttt{uni}(G)\} \cup \rho df$*;*
2. $\Delta'_{\mathsf{P}} \coloneqq \{p \in \texttt{uni}(G) \mid$ *either* (s, p, o)*,* $(s, p, \star_c)$*,* $(\star_c, p, o)$*,* (p, sp, q)*,* (q, sp, p)*,* (p, dom, c)*,* (p, range, d) *or* $(p, \perp_{\mathsf{p}}, q)$ *is in* $\mathsf{Cl}_{\perp}^{\neg}(G)\} \cup \rho df_{\perp}^{\neg}$*;*
3. $\Delta_{\mathsf{P}} \coloneqq \Delta'_{\mathsf{P}} \cup \{\neg p \mid p \in \Delta'_{\mathsf{P}}\}$*;*
4. $\Delta'_{\mathsf{C}} \coloneqq \{c \in \texttt{uni}(G) \mid$ *either* (x, type, c)*,* (c, sc, d)*,* (d, sc, c)*,* (p, dom, c)*,* (p, range, c)*,* $(s, p, \star_c)$*,* $(\star_c, p, o)$ *or* $(c, \perp_{\mathsf{c}}, d)$ *is in* $\mathsf{Cl}_{\perp}^{\neg}(G)\}$*;*

5. $\Delta_{\mathsf{C}} \coloneqq \Delta'_{\mathsf{C}} \cup \{\neg c \mid c \in \Delta'_{\mathsf{C}}\}$;

6. $\Delta_{\mathsf{L}} \coloneqq \mathtt{uni}(G) \cap \mathbf{L}$;

7. $\mathsf{P}^+[\![\cdot]\!]$ *and* $\mathsf{P}^-[\![\cdot]\!]$ *are extension functions* $\Delta_{\mathsf{P}} \to 2^{\Delta_{\mathsf{R}} \times \Delta_{\mathsf{R}}}$ *s.t.* $\mathsf{P}^+[\![p]\!] \coloneqq \{(s,o) \mid (s,p,o) \in \mathsf{Cl}^{\neg}_{\perp}(G)\}$ *and* $\mathsf{P}^-[\![p]\!] \coloneqq \mathsf{P}^+[\![\neg p]\!]$;

8. $\mathsf{C}^+[\![\cdot]\!]$ *and* $\mathsf{C}^-[\![\cdot]\!]$ *are extension functions* $\Delta_{\mathsf{C}} \to 2^{\Delta_{\mathsf{R}}}$ *s.t.* $\mathsf{C}^+[\![c]\!] \coloneqq \{x \in \mathtt{uni}(G) \mid (x, \mathsf{type}, c) \in \mathsf{Cl}^{\neg}_{\perp}(G)\}$ *and* $\mathsf{C}^-[\![c]\!] \coloneqq \mathsf{C}^+[\![\neg c]\!]$;

9. $\cdot^{\mathcal{I}_G}$ *is the identity function over* Δ_{R}.

Then, $\mathcal{I}_G \models^{\perp}_{\neg,} G$.

Proof. We prove that $\mathcal{I}_G$ satisfies the constraints in Definition 5. We illustrate here the proof of some of the conditions in Definition 5 only. The others can be worked out similarly.

Simple:

1. Suppose $(s,p,o) \in G$ and neither s nor o are of the form $\star_c$. Then by construction $p \in \Delta_{\mathsf{P}}$ and $(s,o) \in \mathsf{P}^+[\![p]\!]$, which concludes.
2. Suppose $(s,p,\star_c) \in G$. Then, by construction $p \in \Delta_{\mathsf{P}}$ and $c \in \Delta_{\mathsf{C}}$. Now, assume $y \in \mathsf{C}^+[\![c]\!]$ and, thus, by construction $(y, \mathsf{type}, c) \in \mathsf{Cl}^{\neg}_{\perp}(G)$. Therefore, by rule $(4xe)$ we have also $(s,p,y) \in \mathsf{Cl}^{\neg}_{\perp}(G)$ and, thus, $(s,y) \in \mathsf{P}^+[\![p]\!]$ by construction, which concludes.

Subclass:

2. Assume $(c,d) \in \mathsf{P}^+[\![\mathsf{sc}]\!]$. By construction, $(c, \mathsf{sc}, d) \in \mathsf{Cl}^{\neg}_{\perp}(G)$ and $c, d \in \Delta_{\mathsf{C}}$. Now, assume $x \in \mathsf{C}^+[\![c]\!]$ and, thus, by construction $(x, \mathsf{type}, c) \in \mathsf{Cl}^{\neg}_{\perp}(G)$. Therefore, by rule $(3b)$ we have also $(x, \mathsf{type}, d) \in \mathsf{Cl}^{\neg}_{\perp}(G)$ and, thus, $x \in \mathsf{C}^+[\![d]\!]$ by construction. As a consequence, $\mathsf{C}^+[\![c]\!] \subseteq \mathsf{C}^+[\![d]\!]$. Eventually, assume $x \in \mathsf{C}^-[\![d]\!]$ and, thus, by construction both $x \in \mathsf{C}^+[\![\neg d]\!]$ and $(x, \mathsf{type}, \neg d) \in \mathsf{Cl}^{\neg}_{\perp}(G)$ hold. But, $(c, \mathsf{sc}, d) \in \mathsf{Cl}^{\neg}_{\perp}(G)$ implies, by rule $(3c)$, $(\neg d, \mathsf{sc}, \neg c) \in \mathsf{Cl}^{\neg}_{\perp}(G)$ and, thus, by rule $(3b)$ we have $(x, \mathsf{type}, \neg c) \in \mathsf{Cl}^{\neg}_{\perp}(G)$. Therefore, by construction $x \in \mathsf{C}^+[\![\neg c]\!] = \mathsf{C}^-[\![c]\!]$ and, thus, $\mathsf{C}^-[\![d]\!] \subseteq \mathsf{C}^-[\![c]\!]$, which concludes.

Typing I:

2. Assume $(p,c) \in \mathsf{P}^+[\![\mathsf{dom}^{\mathcal{I}}]\!]$ and $(x,y) \in \mathsf{P}^+[\![p]\!]$. By construction, both (p, dom, c) and (x,p,y) are in $\mathsf{Cl}^{\neg}_{\perp}(G)$. Therefore, by rule $(4a)$, $(x, \mathsf{type}, c) \in \mathsf{Cl}^{\neg}_{\perp}(G)$ and, thus, by construction $x \in \mathsf{C}^+[\![c]\!]$, which concludes.
4. Assume $(p,c) \in \mathsf{P}^+[\![\mathsf{dom}^{\mathcal{I}}]\!]$, $x \in \mathsf{C}^-[\![c]\!]$ and $y \in \mathsf{P}^+[\![p]\!]\downarrow$. By construction, we have that $\{(p, \mathsf{dom}, c), (x, \mathsf{type}, \neg c), (z, p, y)\} \subseteq \mathsf{Cl}^{\neg}_{\perp}(G)$. Therefore, by rule $(4c)$, $(x, \neg p, y) \in \mathsf{Cl}^{\neg}_{\perp}(G)$ and, thus, by construction, $(x,y) \in \mathsf{P}^+[\![\neg p]\!] = \mathsf{P}^-[\![p]\!]$, which concludes.

Typing II:

2. Assume $(p, c) \in \mathsf{P}^+[\![\mathsf{dom}^{\mathcal{I}}]\!]$. Then, by construction $(p, \mathsf{dom}, c) \in \mathsf{Cl}^{\neg}_{\bot}(G)$ and, thus, $p \in \Delta'_{\mathsf{P}} \subseteq \Delta_{\mathsf{P}}$ and $c \in \Delta'_{\mathsf{C}} \subseteq \Delta_{\mathsf{C}}$, which concludes.

Disjointness I:

3. **Symmetry:** Assume $(c, d) \in \mathsf{P}^+[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$. By construction, $(c, \bot_{\mathsf{c}}, d) \in \mathsf{Cl}^{\neg}_{\bot}(G)$ and, thus, by rule $(4a)$ $(d, \bot_{\mathsf{c}}, c) \in \mathsf{Cl}^{\neg}_{\bot}(G)$. Therefore, by construction, $(d, c) \in \mathsf{P}^+[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$, which concludes.

Disjointness II:

1. Assume $(p, c) \in \mathsf{P}^+[\![\mathsf{dom}^{\mathcal{I}}]\!]$, $(q, d) \in \mathsf{P}^+[\![\mathsf{dom}^{\mathcal{I}}]\!]$, and $(c, d) \in \mathsf{P}^+[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$. Then, by construction, we have that $\{(p, \mathsf{dom}, c), (q, \mathsf{dom}, d), (c, \bot_{\mathsf{c}}, d)\} \subseteq \mathsf{Cl}^{\neg}_{\bot}(G)$ and, thus, by rule $(8a)$ $(p, \bot_{\mathsf{p}}, q) \in \mathsf{Cl}^{\neg}_{\bot}(G)$. Therefore, by construction, $(p, q) \in \mathsf{P}^+[\![\bot_{\mathsf{p}}{}^{\mathcal{I}}]\!]$, which concludes.
3. If $(c, d) \in \mathsf{P}^+[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$ then, by construction, $(c, \bot_{\mathsf{c}}, d) \in \mathsf{Cl}^{\neg}_{\bot}(G)$ and, thus, by rule $(6d)$ $(c, \mathsf{sc}, \neg d) \in \mathsf{Cl}^{\neg}_{\bot}(G)$. Therefore, by construction, $(c, \neg d) \in \mathsf{P}^+[\![\mathsf{sc}^{\mathcal{I}}]\!]$. Vice-versa, if $(c, \neg d) \in \mathsf{P}^+[\![\mathsf{sc}^{\mathcal{I}}]\!]$ then by construction, $(c, \mathsf{sc}, \neg d) \in \mathsf{Cl}^{\neg}_{\bot}(G)$ and, thus, by rule $(6e)$ $(c, \bot_{\mathsf{c}}, d) \in \mathsf{Cl}^{\neg}_{\bot}(G)$. Therefore, by construction, $(c, d) \in \mathsf{P}^+[\![\bot_{\mathsf{c}}{}^{\mathcal{I}}]\!]$, which concludes.

□

By Proposition 4, it follows that

Corollary 1. *Every $\rho df^{\neg}_{\bot}$-graph is satisfiable.*

We prove now soundness and completeness of our deduction system for $\rho df^{\neg}_{\bot}$. The proofs are inspired by the analogous ones in [20] for ρdf.

The following proposition is needed for soundness.

Proposition 5 (Soundness)**.** *Let G and H be $\rho df^{\neg}_{\bot}$-graphs and let one of the following statements hold:*

1. *there is a map $\mu : H \to G$;*
2. *$H \subseteq G$;*
3. *there is an instantiation R/R' of one of the rules in Definition 6, such that $R \subseteq G$ and $H = G \cup R'$.*

Then, $G \models^{\bot}_{\neg} H$.

Proof. By Corollary 1 we know that G is satisfiable. So, let

$$\mathcal{I} = \langle \Delta_{\mathsf{R}}, \Delta_{\mathsf{P}}, \Delta_{\mathsf{C}}, \Delta_{\mathsf{L}}, \mathsf{P}^+[\![\cdot]\!], \mathsf{P}^-[\![\cdot]\!], \mathsf{C}^+[\![\cdot]\!], \mathsf{C}^-[\![\cdot]\!], \cdot^{\mathcal{I}} \rangle$$

be a model of G, *i.e.*, $\mathcal{I} \models^{\bot}_{\neg} G$. Therefore, $\mathcal{I}$ satisfies the constraints in Definition 5. We have to prove that $\mathcal{I} \models^{\bot}_{\neg} H$. The proof is split in cases depending on rule applications of which we address here only some of them. The other cases can be shown similarly.

Rule $(4e)$**.** Let $\{(s, p, \star_c), (o, \mathsf{type}, c)\} \subseteq R$ for some $R \subseteq G$, $R' = \{(s, p, o)\}$, obtained via the application of rule $(4e)$, and $H = G \cup R'$. As $\mathcal{I} \models^{\perp}_{\neg,} G$ and $R \subseteq G$, $\mathcal{I} \models^{\perp}_{\neg,} R$ follows. Therefore, $\mathcal{I} \models^{\perp}_{\neg,} (o, \mathsf{type}, c)$ and, thus, $o^{\mathcal{I}} \in \mathsf{C}^{+}[\![c^{\mathcal{I}}]\!]$ follows. But, also $\mathcal{I} \models^{\perp}_{\neg,} (s, p, \star_c)$ and, thus, by condition Simple, case 2. in Definition 5, we have that $(s^{\mathcal{I}}, o^{\mathcal{I}}) \in \mathsf{P}^{+}[\![p^{\mathcal{I}}]\!]$. That is, $\mathcal{I} \models^{\perp}_{\neg,} (s, p, o)$. Hence, from $\mathcal{I} \models^{\perp}_{\neg,} R'$, $\mathcal{I} \models^{\perp}_{\neg,} G$ and $H = G \cup R'$, $\mathcal{I} \models^{\perp}_{\neg,} H$ follows.

Rule $(6d)$**.** Let $(c, \perp_{\mathsf{c}}, d) \in R$ for some $R \subseteq G$, $R' = \{(c, \mathsf{sc}, \neg d)\}$, obtained via the application of rule $(6d)$, and $H = G \cup R'$. As $\mathcal{I} \models^{\perp}_{\neg,} G$ and $R \subseteq G$, $\mathcal{I} \models^{\perp}_{\neg,} R$ follows. Therefore, $\mathcal{I} \models^{\perp}_{\neg,} (c, \perp_{\mathsf{c}}, d)$ and, thus, $(c^{\mathcal{I}}, d^{\mathcal{I}}) \in \mathsf{P}^{+}[\![{\perp_{\mathsf{c}}}^{\mathcal{I}}]\!]$. But, by condition Disjointness II, case 3. in Definition 5 we have that $(c^{\mathcal{I}}, \neg d^{\mathcal{I}}) \in \mathsf{P}^{+}[\![\mathsf{sc}^{\mathcal{I}}]\!]$ and, thus, $(c^{\mathcal{I}}, (\neg d)^{\mathcal{I}}) \in \mathsf{P}^{+}[\![\mathsf{sc}^{\mathcal{I}}]\!]$. Therefore, $\mathcal{I} \models^{\perp}_{\neg,} (c, \mathsf{sc}, \neg d)$. Hence, from $\mathcal{I} \models^{\perp}_{\neg,} R'$, $\mathcal{I} \models^{\perp}_{\neg,} G$ and $H = G \cup R'$, $\mathcal{I} \models^{\perp}_{\neg,} H$ follows.

□

Proposition 6. *Let G and H be $\rho df^{\neg}_{\perp}$-graphs. If $G \models^{\perp}_{\neg} H$ then there is a map $\mu : H \to \mathsf{Cl}^{\neg}_{\perp}(G)$.*

Proof. Consider the canonical model

$$\mathcal{I}_G = \langle \Delta_{\mathsf{R}}, \Delta_{\mathsf{P}}, \Delta_{\mathsf{C}}, \Delta_{\mathsf{L}}, \mathsf{P}^{+}[\![\cdot]\!], \mathsf{P}^{-}[\![\cdot]\!], \mathsf{C}^{+}[\![\cdot]\!], \mathsf{C}^{-}[\![\cdot]\!], \cdot^{\mathcal{I}_G} \rangle$$

of G, as defined in Proposition 4. As $G \models^{\perp}_{\neg} H$, $\mathcal{I}_G \models^{\perp}_{\neg,} H$ follows. Therefore, for each $(s, p, o) \in H$, $p^{\mathcal{I}_G} \in \Delta_{\mathsf{P}}$ and $(s^{\mathcal{I}_G}, o^{\mathcal{I}_G}) \in \mathsf{P}^{+}[\![p^{\mathcal{I}_G}]\!]$. By construction, $p^{\mathcal{I}_G} = p$, and $\mathsf{P}^{+}[\![p^{\mathcal{I}_G}]\!] = \mathsf{P}^{+}[\![p]\!] = \{(t, t') \mid (t, p, t') \in \mathsf{Cl}^{\neg}_{\perp}(G)\}$. Finally, since $(s^{\mathcal{I}_G}, o^{\mathcal{I}_G}) \in \mathsf{P}^{+}[\![p]\!]$, we have that $(s^{\mathcal{I}_G}, p, o^{\mathcal{I}_G}) \in \mathsf{Cl}^{\neg}_{\perp}(G)$, *i.e.*, $(s^{\mathcal{I}_G}, p^{\mathcal{I}_G}, o^{\mathcal{I}_G}) \in \mathsf{Cl}^{\neg}_{\perp}(G)$. Therefore, $\cdot^{\mathcal{I}_G}$ is a map such that $H^{\mathcal{I}_G} \subseteq \mathsf{Cl}^{\neg}_{\perp}(G)$, *i.e.*, a map $\cdot^{\mathcal{I}_G} : H \to \mathsf{Cl}^{\neg}_{\perp}(G)$, which concludes. □

From Proposition 6 we get immediately the following corollary:

Corollary 2. *Let G and H be $\rho df^{\neg}_{\perp}$-graphs. If $G \models^{\perp}_{\neg} H$ then there is a proof of H from G where rule $(1a)$ is used at most once and at the end.*

Eventually, combining previous Propositions 5 and 6, we get soundness and completeness of our deductive system.

Theorem 1 (Soundness & Completeness)**.** *Let G and H be $\rho df^{\neg}_{\perp}$-graphs. Then $G \vdash^{\perp}_{\neg} H$ iff $G \models^{\perp}_{\neg} H$.*

Proof. Concerning soundness, if $G \vdash^{\perp}_{\neg} H$ then, by Proposition 5, $G \models^{\perp}_{\neg} H$. Concerning completeness, if $G \models^{\perp}_{\neg} H$ then, by Proposition 6, H can be obtained from $\mathsf{Cl}^{\neg}_{\perp}(G)$ using rule (1a). Therefore, as $G \vdash^{\perp}_{\neg} \mathsf{Cl}^{\neg}_{\perp}(G)$, $G \vdash^{\perp}_{\neg} H$ follows, which concludes. □

Finally, unlike ρdf (see Proposition 1), the size of the closure of a $\rho df_\bot^\neg$ graph G is $\Theta(|G|^3)$. The upper bound comes from the fact that in a triple (s, p, o), for each s, p and o we may have at most $|G|$ terms, while the lower bound is given by the following example.

Example 7. *It can easily be verified that for* $1 \leq i < j \leq n$ *and* $1 \leq l, k, h \leq n$

$$\{(a_i, \mathsf{type}, c), (a_i, p_1, \star_c), (p_i, \mathsf{sp}, p_j)\} \models^\bot_\neg (a_l, p_k, a_h) \ ,$$

and, thus, the number of triples in the closure is $\Omega(|G|^3)$.

Furthermore, it is not difficult to see that if $\star_c$ terms do not occur in a $\rho df_\bot^\neg$ graph G, then the closure of G remains quadratically upper bounded. As case (i) in Remark 1 also applies to $\rho df_\bot^\neg$, it can be shown that

Proposition 7. *Let* G *and* H *be* $\rho df_\bot^\neg$*-graphs. Then*

1. *the closure of* G *is unique and* $|\mathsf{Cl}_\bot^\neg(G)| \in \Theta(|G|^3)$*;*
2. *if* $\star_c$ *terms do not occur in* G *then* $|\mathsf{Cl}_\bot^\neg(G)| \in \Theta(|G|^2)$*;*
3. *deciding* $G \models^\bot_\neg H$ *is an NP-complete problem;*
4. *if* G *is ground then* $\mathsf{Cl}_\bot^\neg(G)$ *can be determined without using implicit typing rules* (5)*;*
5. *if* H *is ground, then* $G \models^\bot_\neg H$ *if and only if* $H \subseteq \mathsf{Cl}_\bot^\neg(G)$*;*
6. *There is no triple* τ *such that* $\emptyset \models \tau$*.*

Eventually, by Proposition 7 it follows immediately that

Corollary 3. *Let* G *and* H *be two ground* $\rho df_\bot^\neg$ *graphs. Then deciding if* $G \models^\bot_\neg H$ *can be done in time* $O(|H||G|^3)$ *and in time* $O(|H||G|^2)$ *if* $\star_c$ *terms do not occur in* G*.*

4 Conclusions

We have addressed the problem to add negative statements of various form considered as relevant for RDFS by the literature. We have presented a sound and complete deductive system that consists of RDFS rules plus some additional rules to deal with the extra type of triples we allow. The design of the semantics has been such that to preserve features such as the canonical model property and computational attractiveness.

As future work, Corollary 3 tells us that there is still some computational complexity gap w.r.t. ρdf (see Proposition 2), which we would like to reduce as much as possible. In particular, we are going to investigate whether the principles of the method proposed in [20] for ρdf can be adapted to $\rho df_\bot^\neg$ as well. Additionally, we would like to address query answering, in particular to extend our framework to SPARQL and to verify whether and how it impacts w.r.t. $\rho df_\bot^\neg$ graphs.

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